Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Derivatives

Day 10: Finding Derivatives by Applying the Definition

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1 Welcome

Welcome to **Day 3** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on how to find derivatives using the definition.

2 Using The Definition

In the previous two days of this course we've been developing our intuition of what is the derivative. Today we'll focus on how to find the derivatives of some actual functions.

Given a function f, the definition of the derivative we've arrived at is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Today we'll apply this definition to find the derivatives of the following functions:

- $f(x) = x^2$
- $f(x) = \frac{1}{x}$
- $f(x) = x^n$

Finding the derivative of the two first functions are good exercises that serve to show how to apply the definition. The formula for the derivative of the third function is an important result. With that formula in our toolkit, we'll be able to derive any polynomial function.

3 The Derivative of x^2

Let's consider the function given by:

$$f(x) = x^2$$

We'll apply the definition of the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Now, we have that:

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2\Delta x.x + (\Delta x)^2$$

Remember that Δx is just a number, so treat it just like any other variable. Now, introducing this in our definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2\Delta x.x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x.x + (\Delta x)^2}{\Delta x}$$

Now we can factor the Δx in the numerator:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

And this limit is really easy to solve: we simply replace Δx by zero. So, the derivative at point x is:

$$f'(x) = 2x$$

So, reminding ourselves of the geometric meaning of this: the slope of the tangent line to the curve $f(x) = x^2$ at point $(x, f(x))$ is given by $2x$.

From the physical point of view, the function $f'(x)$ outputs the speed of the particle at point x. Isn't this mind-blowing?

4 The Derivative of $\frac{1}{x}$

Here I'll show you a second example of how to apply the definition of the derivative. We consider the function:

$$f(x) = \frac{1}{x}$$

The procedure is the same. We find the term $f(x+\Delta x)$ and try to work the definition algebraically. In this case we have:

$$f(x+\Delta x) = \frac{1}{x+\Delta x}$$

So, the definition of the derivative is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

Performing the fraction sum in the numerator we have that:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{x\Delta x(x+\Delta x)} = -\lim_{\Delta x \to 0} \frac{1}{x(x+\Delta x)}$$

Now, solving the limit we get that:

$$f'(x) = -\frac{1}{x^2}$$

5 The Derivative of x^n

This last example is a little trickier. We'll need a more sophisticated algebraic tool, called the binomial theorem. In case you're not familiar with the binomial theorem, I recommend you this series of videos by KhanAcademy: Binomial Theorem.

Now, let's try to derive the function given by:

$$f(x) = x^n$$

We need to find the term:

$$f(x + \Delta x) = (x + \Delta x)^n$$

This is where we need to use the binomial theorem:

$$(x + \Delta x)^n = x^n + \binom{n}{1} x^{n-1} \Delta x + \binom{n}{2} x^{n-2} (\Delta x)^2 + \ldots + (\Delta x)^n$$

Applying this messy expression in the definition of the derivative we get:

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{x^n + \binom{n}{1} x^{n-1} \Delta x + \binom{n}{2} x^{n-2} (\Delta x)^2 + \ldots + (\Delta x)^n - x^n}{\Delta x}$$

The two terms x^n cancel each other in the numerator and we're left with:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\binom{n}{1} x^{n-1} \Delta x + \binom{n}{2} x^{n-2} (\Delta x)^2 + \ldots + (\Delta x)^n}{\Delta x}$$

Now, notice that each term in the numerator is multiplied by Δx at least once. So, we can divide both the numerator and denominator by Δx to get:

$$f'(x) = \lim_{\Delta x \to 0} \left[\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} \Delta x + \ldots. + (\Delta x)^{n-1} \right]$$

Now, when Δx approaches zero, every term that is multiplied by Δx will also go to zero. And the term we're left with doesn't have a Δx. So:

$$f'(x) = \binom{n}{1} x^{n-1} = n x^{n-1}$$

This formula would prove really useful and you will use it throughout calculus. Another way to write it is:

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

That means that to derive x^n we take the exponent, and multiply it by x to the same exponent minus 1. For example:

$$\frac{d}{dx}(x^3) = 3x^2$$

We can also confirm what we did in the previous example:

$$\frac{d}{dx}(x^2) = 2x$$

A special case (not that special really) is the derivative of a constant. Using this rule we get that the derivative of a constant is zero.

This formula is not only valid when n is a positive integer. It is valid even when n is any real number. We can't prove this here, because we need to know about the chain rule and the derivative of natural logarithm. However, we must know **we can apply this formula for any n.**

6 Exercises

1. Find $f'(x)$ for the following functions applying the definition of the derivative:

   (a) $f(x) = x^3$

   (b) $f(x) = \frac{1}{1+x}$

   (c) $f(x) = \frac{1}{x^2}$

2. Find $f'(x)$ applying the formula $\frac{d}{dx}(x^n) = nx^{n-1}$:

   (a) $f(x) = x^{\frac{3}{2}}$

   (b) $f(x) = x^{-\frac{1}{2}}$

   (c) $f(x) = x^{0.8}$

7 Still To Come

- **Day 4: Derivatives of Trigonometric Functions**
- **Day 5: The Chain Rule**
- **Day 6: The Product Rule and Quotient Rule**
- **Day 7: Implicit Differentiation and Inverse Trigonometric Functions**