Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Derivatives

Day 11: Derivatives of Trigonometric Functions

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Contents

1 Welcome

Welcome to **Day 11** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on the derivatives of trigonometric functions. Specifically, we'll find the derivatives of the functions $f(x) = \sin x$ and $f(x) = \cos x$.

2 The Derivative of $\sin x$

To find the derivative of $\sin x$ we'll use the derivative definition and some trigonometric identities. First of all, let's remember this formula from trigonometry:

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

We'll be using this identity. Now, let's consider the function:

$$f(x) = \sin x$$

Using the definition, the derivative of this function is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Let's work first on the numerator of our limit:

$$\sin(x+\Delta x) - \sin(x)$$

If we apply the trigonometric identity I just mentioned to the first term, with $a = x$ and $b = \Delta x$ we get:

$$\sin(x+\Delta x) = \sin x \cos \Delta x + \sin \Delta x \cos x$$

So, our numerator is:

$$\sin(x+\Delta x) - \sin(x) = \sin x \cos \Delta x + \sin \Delta x \cos x - \sin x$$

Factoring $\sin x$ we get:

$$\sin(x+\Delta x) - \sin(x) = \sin x [\cos \Delta x - 1] + \sin \Delta x \cos x$$

If we plug this into our limit we get:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{\sin x [\cos \Delta x - 1] + \sin \Delta x \cos x}{\Delta x}$$

Using the properties of limits, we can separate this into two limits:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sin x [\cos \Delta x - 1] + \sin \Delta x \cos x}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{\sin x [\cos \Delta x - 1]}{\Delta x} + \lim_{\Delta x \to 0} \frac{\sin \Delta x \cos x}{\Delta x}$$

Now, to do everything clearly, we'll solve each of those limits above separately. Let's give them names:

$$A = \lim_{\Delta x \to 0} \frac{\sin x [\cos \Delta x - 1]}{\Delta x}$$

$$B = \lim_{\Delta x \to 0} \frac{\sin \Delta x \cos x}{\Delta x}$$

So, we have:

$$f'(x) = A + B$$

Let's find the value of A first. We'll be using some of our skills with trigonometric limits. First of all, as $\sin x$ does not depend on Δx, we can take it out of the limit sign:

$$A = \sin x \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x}$$

Now we'll use a little algebraic trick to change the sign in the numerator of the fraction. We will invert the signs of the numerator and put a minus sign outside:

$$A = \sin x \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = -\sin x \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x}$$

Just perform that multiplication to check that is the same as what we had before. We did that trick so we can use a trigonometric identity. Now, let's multiply both the numerator and denominator by $1 + \cos \Delta x$:

$$A = -\sin x \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} . \frac{1 + \cos \Delta x}{1 + \cos \Delta x}$$

If we perform the products we get:

$$A = -\sin x \lim_{\Delta x \to 0} \frac{1 - \cos^2 \Delta x}{\Delta x (1 + \cos \Delta x)}$$

Now we use the basic trigonometric identity that relates $\cos \Delta x$ and $\sin \Delta x$:

$$1 - \cos^2 \Delta x = \sin^2 \Delta x$$

The left side of this equation is what we have in the numerator of our limit. So:

$$A = -\sin x \lim_{\Delta x \to 0} \frac{\sin^2 \Delta x}{\Delta x (1 + \cos \Delta x)}$$

You may get a feel of what we are trying to do. We are trying to use the fundamental trigonometric limit (you can learn about it here). We can write A as:

$$A = -\sin x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} . \lim_{\Delta x \to 0} \frac{\sin \Delta x}{1 + \cos \Delta x}$$

The first limit is the fundamental trigonometric limit, and it is equal to 1. The second limit is zero, because $\sin \Delta x$ goes to zero and $1 + \cos \Delta x$ goes to 2. So,

$$A = -\sin x .1.0 = 0$$

Now, you may begin to worry about solving limit B, but it is much easier than this one. We have:

$$B = \lim_{\Delta x \to 0} \frac{\sin \Delta x \cos x}{\Delta x} = \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} =$$

$$= \cos x .1 = \cos x$$

So, finally:

$$f'(x) = \frac{d}{dx}(\sin x) = A + B = 0 + \cos x = \cos x$$

3 The Derivative of $\cos x$

We will now follow a very similar process to prove that the derivative of $\cos x$ is $-\sin x$.

To review this proof will help you understand both proofs better. They're very similar. And you'll also be prepared in case you need to know them for an exam.

Again, we'll use the definition of the derivative:

$$f(x) = \cos x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x}$$

Now, we need to remember this formula from trigonometry:

$$\cos(x+\Delta x) = \cos x \cos \Delta x - \sin x \sin \Delta x$$

So, let's work on our numerator:

$$\cos(x+\Delta x) - \cos x = \cos x \cos \Delta x - \sin x \sin \Delta x - \cos x$$

Factoring $\cos x$:

$$\cos(x+\Delta x) - \cos x = \cos x [\cos \Delta x - 1] - \sin x \sin \Delta x$$

If we divide both sides by Δx we get:

$$\frac{\cos(x+\Delta x) - \cos x}{\Delta x} = \frac{\cos x [\cos \Delta x - 1]}{\Delta x} - \frac{\sin x \sin \Delta x}{\Delta x}$$

Now, we take the limit as $\Delta x \to 0$ of both sides to find the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} =$$

$$\cos x. \lim_{\Delta x \to 0} \frac{[\cos \Delta x - 1]}{\Delta x} - \sin x. \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

The first limit is the one we already calculated (A) and the second one is the fundamental trigonometric limit, which equals 1. So:

$$f'(x) = \frac{d}{dx}(\cos x) = 0 - \sin x.1 = -\sin x$$

And these two formulas are enough to derive all trigonometric functions. We'll do that after we learn about the product rule, the chain rule and the quotient rule.

4 Still To Come

- **Day 5: The Chain Rule**
- **Day 6: The Product Rule and Quotient Rule**
- **Day 7: Implicit Differentiation and Inverse Trigonometric Functions**