INTUITIVE-CALCULUS.COM PRESENTS

The Free Intuitive Calculus Course

Derivatives

Day 12: The Chain Rule

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1 Welcome

Welcome to **Day 12** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on what could be considered the most important computational tool for derivatives: **the chain rule.**

2 Introduction

The chain rule is one of the most important rules in calculus. To derive most interesting functions we'll need it. This rule tells us what is the derivative of a composite function.

Let's say you want to derive the following function:

$$h(x) = \sin x^2$$

This is the composition of two functions: the sin function and the $()^2$ function. To make the meaning of the chain rule clear, we'll start now to use a common notation for composite functions. Let's denote the "squaring" function as:

$$S(x) = x^2$$

So, using this notation, function h is defined by:

$$h(x) = \sin [S(x)]$$

Ok, now the fun part. How do we derive this function. The chain rule is the answer to this question. This rules says that if we have a composite function like this:

$$h(x) = f[g(x)]$$

Then, its derivative is:

$$h'(x) = f'[g(x)]\, g'(x)$$

That is, it is the product of the derivatives of the functions. But, the derivative of f is "evaluated" at $g(x)$. We can think of f as the "outer" function.

Going back to our example, we had the function:

$$h(x) = \sin [S(x)]$$

So, the outer function f in our formula is sin. The inner function g is the squaring function S. Applying the chain rule we have its derivative:

$$h'(x) = \sin' [S(x)]\, S'(x)$$

As we learned in the previous chapter, the derivative of sin is cos. Also, the derivative of x^2 is $2x$. So:

$$h'(x) = \cos(x^2).2x = 2x \cos(x^2)$$

Here we used a special symbol, S, for the "squaring" function. This is not usually done.

To solve chain rule problems it is useful to think about an “outer” function and an “inner” function. Let’s clarify that with an example.

3 A First Example

$$h(x) = \sin\frac{1}{x}$$

We want to find the derivative of this function.

As in the previous example, we have the outer function, *sin*. The inner function is $\frac{1}{x}$. Let's repeat what the chain rule says:

$$h(x) = f[g(x)] \quad\Rightarrow\quad h'(x) = f'[g(x)]\,g'(x)$$

So, what the rule is saying is that to derive a composite function we must:

1. Derive the outer function and evaluate it at $g(x)$.
2. Derive the inner function and evaluate it at x.
3. Multiply these two.

In this case the outer function is $f = \sin$, and the inner function is $g(x) = \frac{1}{x}$. So:

$$f'[g(x)] = \cos[g(x)] = \cos\frac{1}{x}$$

$$g(x) = \frac{1}{x} = x^{-1} \quad\Rightarrow\quad g'(x) = -\frac{1}{x^2}$$

So, our derivative is:

$$h'(x) = f'[g(x)]\,g'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$$

4 A Second Example

What happens if we have:

$$f(x) = \sin^2 x^2$$

We can also write this function as:

$$f(x) = \left(\sin x^2\right)^2$$

If we use again our invented notation for the squaring function:

$$S(x) = x^2$$

This composite function can be written in a convoluted way:

$$f(x) = S[\sin[S(x)]]$$

This may seem complicated but it helps to see how we should apply the chain rule. To use the chain rule, we can give a name to the inner function, for example:

$$g(x) = \sin[S(x)]$$

So, our $f(x)$ is:

$$f(x) = S[\sin[S(x)]] = S[g(x)]$$

Now it is more clear how we should apply the chain rule. We have that:

$$f'(x) = S'[g(x)]\, g'(x)$$

And we know that:

$$S'(x) = \frac{d}{dx}(x^2) = 2x$$

Evaluating this at $g(x)$:

$$S'[g(x)] = 2g(x) = 2\sin[S(x)] = 2\sin x^2$$

We still need to find $g'(x)$, so we apply the chain rule again:

$$g(x) = \sin x^2 \quad \Rightarrow \quad g'(x) = 2x\cos x^2$$

So, finally:

$$f'(x) = S'[g(x)]\, g'(x) = 2\sin x^2 . 2x\cos x^2 = 4x\sin x^2 \cos x^2$$

As you can see, the chain rule can be used even when we have the composition of more than two functions.

5 Faster Solving

In the previous examples we solved the derivatives in a rigorous manner. We applied the formula directly.

Solving derivatives like this you'll rarely make a mistake. But there is a faster way. In fact, this faster method is how the chain rule is usually applied.

Let's try it with example 2. We had:

$$f(x) = \sin^2 x^2 = \left(\sin x^2\right)^2$$

First of all, let's derive the outermost function: the "squaring" function outside the brackets. To do this, we imagine that the function inside the brackets is just a variable y:

$$f(x) = \left(\sin x^2\right)^2 = y^2, \text{ (where } y = \sin x^2 \text{)}$$

And I say imagine because you don't need to write it like this! If it were just a y we'd have:

$$f'(x) = 2y.(\quad)$$

But y is a function. Inside the empty parenthesis, according the chain rule, we must put the derivative of y.

Replacing y by its real value:

$$f'(x) = 2\sin x^2 \sin' x^2$$

With practice you'll be able to do all this in your head. Now, we only need to derive the inside function $\sin x^2$:

$$\sin' x^2 = 2x\cos x^2$$

So, our derivative is:

$$f'(x) = 2\sin x^2 .2x\cos x^2 = 4x\sin x^2 \cos x^2$$

6 Exercises

These exercises can be challenging if this is the first time you are exposed to the chain rule. For that reason, it is very important for you to do them. If you get lost, try to review what the chain rule says and the examples.

Apply the chain rule to find the derivative of:

1. $f(x) = a\sqrt{\cos 2x}$
2. $f(x) = \tan x$
   Hint: *Use the identity* $\tan^2 x = \frac{1}{\cos^2 x} - 1$ *and derive both sides of the equation.*
3. $f(x) = \sin(\sin x)$

7 Still To Come

- **Day 13: The Product Rule and Quotient Rule**
- **Day 14: Implicit Differentiation and Inverse Trigonometric Functions**