Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Derivatives

Day 13: The Product Rule and Quotient Rule

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1 Welcome

Welcome to **Day 13** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on two additional rules that can be used to find derivatives: **the product rule** and **the quotient rule.**

2 The Product Rule

As you may expect, the product rule tells you how to derive the product of two functions. For example, the function given by:

$$f(x) = x \sin x$$

This function is the product of the two functions $u(x) = x$ and $v(x) = \sin x$. In the case of limits, the limit of the product of two functions is the product of the limits of each function. This is not the case with derivatives.

The product rule says that if $f(x) = u(x)v(x)$, then its derivative is:

$$f'(x) = u(x)v'(x) + u'(x)v(x)$$

This rule may seem strange at first, but you get used to it pretty quickly. The proof of this rule is really simple, it only involves the definition of the derivative and an algebraic trick.

Suppose we have a function given by $f(x) = u(x)v(x)$. By definition, its derivative is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x}$$

The algebraic trick I just mentioned is to add and subtract the expression $u(x)v(x + \Delta x)$ to the numerator. If we do that, we get:

$$f'(x) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x + \Delta x) + u(x)v(x + \Delta x) - u(x)v(x)}{\Delta x}$$

Factoring things in the numerator we have that:

$$f'(x) = \lim_{\Delta x \to 0} \frac{v(x + \Delta x)\left[(u(x + \Delta x) - u(x)\right] + u(x)\left[v(x + \Delta x) - v(x)\right]}{\Delta x}$$

We do that so we can separate this fraction in two:

$$f'(x) = \lim_{\Delta x \to 0} v(x + \Delta x)\frac{u(x + \Delta x) - u(x)}{\Delta x} + u(x) \lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x}$$

In the first limit, $v(x + \Delta x)$ approaches $v(x)$ because we consider the function v continuous (if you aren't very comfortable with this notion read this page: Continuous Functions).

In the second limit, $u(x)$ does not depend on Δx, that's why we've taken it out of the limit sign. The two remaining limits are the definitions of $v'(x)$ and $u'(x)$. So, we're left with the formula:

$$f'(x) = u(x)v'(x) + u'(x)v(x)$$

3 Some examples

We'll start with a simple function:

$$f(x) = x \sin x$$

Here we have that:

$$u(x) = x, \qquad v(x) = \sin x$$

And the derivatives of these are:

$$u'(x) = 1, \qquad v'(x) = \cos x$$

So, applying the product rule:

$$y' = u(x)v'(x) + v(x)u'(x) = x \cos x + \sin x$$

Easy enough.

Let's see what would had happened if we simply derived and multiplied, as we suggested at the beginning of this page:

$$u'(x)v'(x) = 1. \cos x = \cos x \neq y'!!$$

Very different result. I'm showing you this to make sure you won't make this mistake.

4 Product of Three Functions

Now, this one is a little harder:

$$y = x \sin x \cos x$$

We now have the product of three functions. What do we do?

Always remember, something that looks difficult always is the combinations of many easy things. Let's tackle this in easy chunks. We can do the following:

$$y = u(x)v(x)$$

$$\text{where } u(x) = x \sin x, \quad v(x) = \cos x$$

Now we can simply apply the product rule:

$$y' = u(x)v'(x) + u'(x)v(x)$$

We now need to find $u'(x)$ and $v'(x)$:

$$u(x) = x \sin x, \text{ applying the product rule here again } \Rightarrow u'(x) = x \cos x + \sin x$$

$$v(x) = \cos x \quad \Rightarrow \quad v'(x) = -\sin x$$

So, replacing in the formula we have that:

$$y' = x \sin x.(-\sin x) + (x \cos x + \sin x).\cos x$$

$$= -x \sin^2 x + x \cos^2 x + \sin x \cos x$$

5 The Quotient Rule

The quotient rule is a rule that tells you what is the derivative of the quotient of two functions. I never memorized it, even after teaching it for a few years. That is because I almost never use it.

Why don't I use it? Because the quotient rule is just a special case of the product rule. You can always transform a quotient into a product. Now that we also know the chain rule, we don't need to memorize another rule.

Let me show you how I solve problems involving quotients without applying the quotient rule. For example, consider the function:

$$f(x) = \frac{x^3}{\cos x}$$

If you are able to remember the formula for quotients found in textbooks you can simply apply the formula. I assure you, though, that you won't remember it one year from now.

There's another way to solve this. We can apply the product rule. First, let's write this expression as a product:

$$f(x) = \frac{x^3}{\cos x} = x^3.(\cos x)^{-1}$$

Now we have a product. That means we can apply the product rule. First we determine our functions u and v:

$$f(x) = u(x).v(x)$$

$$\text{where } u(x) = x^3, \quad v(x) = (\cos x)^{-1}$$

And we know how to derive this:

$$f'(x) = u(x)v'(x) + u'(x)v(x)$$

We have that:

$$u(x) = x^3 \quad \Rightarrow \quad u'(x) = 3x^2$$

Now applying the chain rule:

$$v(x) = (\cos x)^{-1} \quad \Rightarrow \quad v'(x) = (-1)(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$$

And with some algebra we get that:

$$f'(x) = x^3 . \frac{\sin x}{\cos^2 x} + 3x^2 . (\cos x)^{-1}$$

$$f'(x) = \frac{x^3 \sin x}{\cos^2 x} + \frac{3x^2}{\cos x}$$

And that's it. I really think it would be much better for you to learn how to solve problems like this. The only requisite is to be a master chain rule technician. And I'm sure you already are one.

6 Exercises

Find $f'(x)$ for the functions:

1. $f(x) = \frac{2x^4}{b^2 - x^2}$
2. $f(x) = \sin 2x \cos 3x$
3. $f(x) = \frac{\sin x}{1 + \cos x}$

7 Still To Come

- **Day 14: Implicit Differentiation and Inverse Trigonometric Functions**