Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Derivatives

Day 14: Implicit Differentiation and Inverse Trigonometric Functions

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1 Welcome

Welcome to **Day 14** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on a technique known as **implicit differentiation** and also on **inverse trigonometric functions.**

2 Implicit Functions

With implicit differentiation we try to find the derivatives of what are called implicit functions. But what are implicit functions? A good starting point is to learn what they are not.

Until now we've been deriving functions that are not implicit, we could call them explicit. For example, consider the function defined by the equality:

$$f(x) = x^2$$

This is called explicit because given an x, you can directly get $f(x)$ by replacing the value of x in the right side of the equation.

What if we have a relation like this one:

$$x^2 + y^2 = 10$$

This relation defines a function. In fact, it defines more than one function. But we are only interested in one right now.

Let's say you give some value to variable x, say $x = 2$. So, replacing that value in the equation I can get:

$$x^2 + y^2 = 10$$

$$2^2 + y^2 = 10$$

$$y^2 = 10 - 2^2 = 10 - 4 = 6$$

$$y = \pm\sqrt{6}$$

We can get the value of y. So, y is a function of x. However, it wasn't as easy to get the value of y in this case, compared to the case where we have $f(x) = x^2$.

This is why we call this type of function implicit. The function is kind of hidden in the equation.

On the other hand, with explicit functions you don't have to do any work to get y, or $f(x)$.

3 A First Example

Let's stick with the previous example. We have the equation:

$$x^2 + y^2 = 10$$

We know there's a function y hidden in there. And let's say we want to find the derivative of that function.

So, first of all, to make clear that y is a function of x, let's change the symbol y for the symbol $f(x)$. That is: $y = f(x)$.

Using that notation, we can write our original equation like this:

$$x^2 + [f(x)]^2 = 10$$

It would be useful to make a clear distinction now. It is about notation. When we write the symbol $\frac{dy}{dx}$ we're referring to the derivative of y with respect to x. This symbol is a noun. It represents the derivative (a function).

On the other hand, when we write the symbol $\frac{d}{dx}$ we're using it as a verb. We refer to the operation of taking the derivative. It's like a commandment: *take the derivative of what follows with respect to x*. We'll call this symbol the derivative operator.

Keeping that in mind, let's return to our equation:

$$x^2 + [f(x)]^2 = 10$$

We can think about this equation as the equality between two functions of x. If these two functions are equal, their derivatives must be equal too. Just common sense.

So, we take the derivative of both sides of the equation and make them equal:

$$\frac{d}{dx}\left[x^2 + [f(x)]^2\right] = \frac{d}{dx}(10)$$

In the left side we can "distribute" the derivative:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}[f(x)]]^2 = \frac{d}{dx}(10)$$

And we know that the derivative of 10(a constant) is zero. The first derivative in the left side is easy:

$$2x + \frac{d}{dx}[f(x)]^2 = 0$$

Now, let's take a look at the derivative that is left in the previous equation. It is the derivative with respect to x of the function:

$$[f(x)]^2$$

This is a composite function, that first applies the function f to x and then squares it. So, the outer function is the squaring function and the inner function is the function f.

So, applying the chain rule, we first take the derivative of the outer function:

$$\frac{d}{dx}[f(x)]^2 = 2f(x).(\quad)$$

And then multiply that by the derivative of the inner function:

$$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$$

The derivative of the inside is an unknown. It is what we were originally looking for, remember? So, we just use the symbol $f'(x)$ to represent it. The chain rule holds true, so the above equality is true.

Replacing it in our equation we get that:

$$2x + 2f(x)f'(x) = 0$$

Dividing both sides of the equation by 2 and solving for $f'(x)$ we get:

$$f'(x) = -\frac{x}{f(x)}$$

If we now return to our original notation: $y = f(x)$ we get that:

$$y' = -\frac{x}{y}$$

And that's the answer. If we want the answer to be only a function of x we can solve for y in the original equation.

Notice that you could have arrived to this solution using explicit differentiation. That's because for this particular equation you could have solved for y. There are many cases where that isn't possible, though.

4 Inverse Trigonometric Functions

Using implicit differentiation we'll be able to find the derivatives of inverse trigonometric functions. That is, the functions given by:

$$f(x) = \arcsin x$$

$$g(x) = \arccos x$$

$$h(x) = \arctan x$$

I'll show you the process with the first one. For the others, I'll leave you the steps for you to work it out.

By definition, the function $f(x) = \arcsin x$ satisfies the equality:

$$\sin(\arcsin x) = x$$

This is just the definition of $\arcsin x$. Now, in the left of the previous equation we have a composite function, so we can derive it using the chain rule. To the right we have a very simple function. So, applying implicit differentiation, we have that:

$$\frac{d}{dx}[\sin(\arcsin x)] = \frac{d}{dx}(x)$$

Applying the chain rule:

$$\cos(\arcsin x).\frac{d}{dx}(\arcsin x) = 1$$

Now, dividing both sides of the previous equation by $\cos(\arcsin x)$ we get that:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)}$$

Now, we'll arrive to a more useful expression for this derivative. There is a well known trigonometric identity that tells that:

$$\cos y = \sqrt{1 - \sin^2 y}$$

So, using $y = \arcsin x$ we have that:

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)}$$

Now:

$$\sin^2(\arcsin x) = \sin(\arcsin x).\sin(\arcsin x) = x.x = x^2$$

So, we have that:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

This derivative is usually found on most tables of derivatives. However, it is useful to know where it came from.

5 Exercises

1. Prove that $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ following these steps:

   (a) Derive both sides of the equation:

   $$\cos(\arccos x) = x$$

   (b) Solve for $\frac{d}{dx}(\arccos x)$.

   (c) Use the trigonometric identity:

   $$\sin(\arccos x) = \sqrt{1 - \cos^2(\arccos x)}$$

2. Prove the same formula first proving the fact that $\arccos x = \frac{\pi}{2} - \arcsin x$.

3. Prove that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.