Intuitive-Calculus.com Presents

# The Free Intuitive Calculus Course

## Integrals

## Day 15: Introduction to Indefinite Integrals

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# Contents

# 1 Welcome

Welcome to **Day 15** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we begin to study **integrals.** This part of the course is divided into the following days:

- **Day 15: Introduction to Indefinite Integrals**
- **Day 16: Solving Integrals by Substitution**
- **Day 17: Integration by Parts**
- **Day 18: Integrals by Trigonometric Substitution**
- **Day 19: Solving Trigonometric Integrals**
- **Day 20: Introduction to Definite Integrals**
- **Day 21: The Fundamental Theorem of Calculus**

# 2 What Are Indefinite Integrals

Our goal is to learn what are definite integrals and how to calculate them. Definite integrals are the generalization of the concept of **area.** In fact, they allow us to calculate areas of many complex figures.

There are many applications of definite integrals besides the calculation of areas, but presenting them as areas is the the simplest and most intuitive way of first introducing the concept.

In order to learn about definite integrals, we need to learn first about **indefinite integrals.** The concept of indefinite integral is very simple. When we learned about derivatives you were supposed to solve the following problem:

**Problem 1**

*Given the function $f(x)$, find the function $f'(x)$.*

Now, we are interested in solving the inverse problem:

**Problem 2**

*Given the function $f(x)$, find a function $F(x)$ such that $F'(x) = f(x)$.*

That is, given a function $f(x)$, we want to find a function $F(x)$ whose derivative is $f(x)$. We call such a function $F(x)$ a ***primitive*** of $f(x)$. Let's see a concrete example and you'll realize that in some cases this problem isn't that hard.

Let's consider the function:

$$f(x) = 2x$$

We know that the derivative of the function $F(x) = x^2$ is $F'(x) = 2x$. So, we solved the problem, $F(x)$ is a ***primitive*** of $f(x)$.

However, we have another little problem here. What if we want to find **all** the primitives of $f(x)$? Let's consider the functions:

$$F_1(x) = x^2 + 1$$

$$F_2(x) = x^2 + 2$$

We also have that $F_1'(x) = F_2'(x) = 2x = f(x)$. So, they are also primitives of $f(x)$. In general, any function of the form:

$$F(x) = x^2 + C$$

where $C$ is a constant, is a primitive of $f(x)$. The interesting thing is that, all the primitives of the function $f(x)$ would be as $F(x)$.

That is, to find all the primitives of a function $f(x)$, we only need to find one primitive. All the others would be the same primitive plus a constant. This is the content of the following theorem, which is a consequence of the *Mean Value Theorem* for derivatives:

**Theorem 1**

*If $F(x)$ and $G(x)$ are such that $F'(x) = G'(x) = f(x)$ for all $x$ in their domains, there exists a constant $C$ such that $G(x) = F(x) + C$.*

*Proof.* Let's define the function $H(x) = G(x) - F(x)$. We have that $H'(x) = G'(x) - F'(x) = 0$. Given $x$ and $y$, we have, by the Mean Value Theorem, that there is a $c$ between $x$ and $y$ such that:

$$\frac{H(y) - H(x)}{y - x} = H'(c)$$

But $H'(c) = 0$. So, that means that the numerator of the fraction must be zero:

$$H(y) - H(x) = 0 \Rightarrow H(y) = H(x)$$

As this is true for any $x$ and $y$, we have that $H$ is constant. That is, we can write:

$$H(x) = G(x) - F(x) = C$$

$$G(x) = F(x) + C$$

where $C$ is a constant. □

We are going to define the ***indefinite integral*** of a function $f(x)$ as the set of all primitives of that function. To denote the indefinite integral of a function $f(x)$ we use the notation:

$$\int f(x)dx$$

This is read: *the indefinite integral of $f$ of $x$, dee $x$.* We use $dx$ to specify the variable. The usefulness of this will become obvious in the following lessons.

Theorem 1 tells us that, given a primitive of a function $f(x)$, we can find all the other primitives by adding constants. That's why, if $F(x)$ is a primitive of $f(x)$, we are going to write:

$$\int f(x)dx = F(x) + C$$

For example, we already know that:

$$\int 2x dx = x^2 + C$$

# 3 Another Example

We already calculated our first integral:

$$\int 2x dx = x^2 + C$$

We calculated this integral by simply realizing that $\left(x^2\right)' = 2x$. And this is generally how we find indefinite integrals. That is, we use our knowledge of derivatives of functions. Let's see another example:

**Example 1**

Find the integral:

$$\int \frac{1}{x} dx$$

*Solution.* We usually use the notation:

$$\int \frac{dx}{x} = \int \frac{1}{x} dx$$

What we need to do to find this integral is to find a primitive of the function $f(x) = \frac{1}{x}$. We know that:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So, the function $f(x) = \ln x$ is a primitive. Then, we can write:

$$\boxed{\int \frac{dx}{x} = \ln x + C}$$

# 4 A Short Table of Integrals

Now, we are ready to build a short table of integrals. You can check each of the formulas by simply taking the derivative of the function on the right side of each equation.

**Table of Integrals**

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{dx}{x} = \ln x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{1+x^2} = \arctan x + C$$

# 5 Some Properties of Integrals

Here are some properties of integrals that can be useful when you are trying to solve problems.

**Properties of Integrals**

1. The integral of a derivative is:

$$\int f'(x)dx = f(x) + C$$

   Just remember the definition of *primitive* of $f'(x)$: a function whose derivative is $f'(x)$. That's $f(x)$!

2. The integral of the sum is the sum of the integrals:

$$\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$$

   This property is checked using the analogous property of derivatives.

3. You can take constants out of the integral sign:

$$\int Af(x)dx = A\int f(x)dx$$

   if $A$ is a constant. This property is also checked using the analogous property of derivatives.

# 6 Exercises

Solve the following integrals:

1. $\int \sqrt{x} dx$
2. $\int \left( \frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{2} \right) dx$
3. $\int \left( x^2 + \frac{1}{\sqrt[3]{x}} \right)^2 dx$

# 7 Still To Come

- **Day 16: Solving Integrals by Substitution**
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