Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Integrals

Day 16: Integration by Substitution

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Contents

1 Welcome

Welcome to **Day 16** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we learn our first technique for solving integrals: substitution.

2 The Intuitive Idea

The technique we are going to learn has several names: some call it ***u-substitution.*** The idea is to use the chain rule in reverse. Let's say we want to calculate the integral:

$$\int \sin^2 x \cos x dx$$

We know that the derivative of $\sin x$ is $\cos x$. So, we are going to make the change of variables:

$$u = \sin x$$

The integral of this new function with respct to x is:

$$\frac{du}{dx} = \cos x$$

So, we substitute the variable u for $\sin x$ and $\frac{du}{dx}$ for $\cos x$ in our integral:

$$\int \sin^2 x \cos x dx = \int u^2 \frac{du}{dx}.dx$$

Now, you may not be comfortable with the notation dx, but we can treat this *differential* as a number. Specifically, we can *cancel* the two dx in our integral:

$$\int u^2 \frac{du}{\cancel{dx}}.\cancel{dx} = \int u^2 du$$

If you want an intuitive explanation of why this can be done, please read this page.

Now, this integral we know how to solve:

$$\int u^2 du = \frac{u^3}{3} + C$$

The only thing left to do is to substitute back our original variable ($u = \sin x$):

$$\int \sin^2 x \cos x dx = \frac{\sin^3 x}{3} + C$$

You may check that this answer is correct by taking the derivative of the right side. You should get the function we are integrating.

Now, why did I say that this is like using the chain rule in reverse? Because this method was based on the recognition of a special *model* of integral. That is, the integral we solved is in the form:

$$\int g'[f(x)]f'(x)dx$$

In our specific example:

$$g(y) = y^2, \qquad f(x) = \sin x$$

The chain rule says that the derivative of a composite function is:

$$\frac{d}{dx}[g(f(x))] = g'[f(x)]f'(x)$$

The function we are trying to integrate is in the form of the right side of the equation. So, the chain rule says that a primitive of this function is the composite function $g[f(x)]$. And this is precisely what we got as the answer of our problem.

3 More Examples

Example 1

Find the integral:

$$\int \frac{\ln x}{x} dx$$

Solution. We know that the derivative of $f(x) = \ln x$ is $f'(x) = \frac{1}{x}$. So, we make the substitution:

$$u = \ln x$$

Now, we need to find dx. To do that, we take the derivative of u:

$$\frac{du}{dx} = \frac{1}{x}$$

Making the substitution we get:

$$\int \frac{\ln x}{x} dx = \int u.\frac{du}{dx}.dx = \int u.\frac{du}{\cancel{dx}}.\cancel{dx}$$

$$= \int u du = \frac{u^2}{2} + C$$

Substituting back our original variable:

$$\boxed{\int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C}$$

Notice that we always make the substitution $u = f(x)$, when we think of these integrals as in the form:

$$\int g'[f(x)] f'(x) dx$$

Let's see more examples:

Example 2

Find the integral:

$$\int \sin ax dx$$

Solution. In this case, we make the substitution:

$$u = ax$$

The derivative is:

$$\frac{du}{dx} = a$$

If we divide both sides by a in this equation:

$$\frac{1}{a}\frac{du}{dx} = 1$$

We do this because we don't have an a in our integral. In this way we can substitute:

$$\int \sin ax dx = \int \sin ax.1 dx = \int \sin u.\frac{1}{a}.\frac{du}{dx}.dx = \frac{1}{a}\int \sin u.\frac{du}{\cancel{dx}}.\cancel{dx}$$

$$= \frac{1}{a}\int \sin u du = \frac{1}{a}.(-\cos u) + C$$

Substituting back:

$$\boxed{\int \sin ax dx = -\frac{\cos ax}{a} + C}$$

Example 3

Find the integral:

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

Solution. First of all, we note that we can write this as:

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx$$

So, we make the substitution:

$$u = e^x, \qquad \frac{du}{dx} = e^x$$

We are going to put the variable u in the denominator and $\frac{du}{dx}$ in the numerator:

$$\int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{1}{1+u^2}.\frac{du}{dx}.dx = \int \frac{1}{1+u^2}.\frac{du}{\cancel{dx}}.\cancel{dx} = \int \frac{du}{1+u^2}$$

Now, we need to look back at our little table of integrals that we saw yesterday, to see that this integral is:

$$\int \frac{du}{1+u^2} = \arctan u + C$$

As always, you can check this by taking the derivative of the right side. Now, substituting back:

$$\boxed{\int \frac{e^x}{1+e^{2x}}\,dx = \arctan e^x + C}$$

4 Exercises

Solve the following integrals:

1. $\int e^{3x} dx$
2. $\int \frac{dx}{1+x}$
3. $\int \frac{dx}{5-x}$
4. $\int \tan 2x dx$
5. $\int \sqrt{x^2+1} x dx$
6. $\int \frac{\sin 2x dx}{\sqrt{1+\sin^2 x}}$

5 Still To Come

- **Day 17: Integration by Parts**
- **Day 18: Integrals by Trigonometric Substitution**
- **Day 19: Solving Trigonometric Integrals**
- **Day 20: Introduction to Definite Integrals**
- **Day 21: The Fundamental Theorem of Calculus**