Intuitive-Calculus.com Presents

# The Free Intuitive Calculus Course

## Integrals

## Day 17: Integration Parts

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# 1 Welcome

Welcome to **Day 17** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we learn a very important technique for solving integrals: integration by parts.

# 2 The Idea

We already know how to solve integrals by substitution. We saw that what is behind integration by substitution is the chain rule. That is, we use the chain rule in reverse. What we are going to try to do now is to use the ***product rule*** in reverse.

The product rule says that if you have two functions, $u(x)$ and $v(x)$, then, the derivative of their product equals:

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

Now, let's take the indefinite integral of both sides:

$$\int [u(x)v(x)]'\,dx = \int [u'(x)v(x) + u(x)v'(x)]\,dx$$

These integrals are equal because the functions we are integrating are equal. Now, the integral of the sum, equals the sum of the integrals:

$$\underbrace{\int [u(x)v(x)]'\,dx}_{u(x)v(x)} = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

Let's look a bit closer at the integral on the left side. We have the integral of a derivative. We know that the integral of a derivative is the function we are differentiating, so:

$$u(x)v(x) = \int u'(x)v(x)dx + \underbrace{\int u(x)v'(x)dx}_{\text{let's solve for this!}}$$

And let's solve for the integral that is underlined:

$$\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx}$$

This is the formula of ***integration by parts***. When you say you are going to use integration by parts, you mean that you are going to solve the integral on the left by using this formula.

This formula is usually useful because the integral on the right side of the equation is usually easier to solve. Let's see some examples...

# 3 Examples

**Example 1**

Find the integral:

$$\int xe^x dx$$

*Solution.* This integral can't be solved using simple substitution. So, let's try integration by parts. Our formula says:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

We are going to define our variables:

$$u(x) = x, \qquad v'(x) = e^x$$

In the formula we also have $u'(x)$ and $v(x)$, so we need to find these:

$$u'(x) = 1, \qquad v(x) = e^x$$

To find $v(x)$ we simply take a primitive of $v'(x)$. In this case, it happens to be the same function because $(e^x)' = e^x$. Now, substituting in our formula:

$$\int \underbrace{x}_{u(x)} . \underbrace{e^x}_{v'(x)} .dx = \underbrace{x}_{u(x)} . \underbrace{e^x}_{v(x)} - \int \underbrace{1}_{u'(x)} . \underbrace{e^x}_{v(x)} .dx$$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

$$\boxed{\int xe^x dx = e^x(x-1) + C}$$

As always, you can check this answer by taking the derivative of the right side.

Example 2

Find the integral:

$$\int x \sin x dx$$

*Solution.* First of all, we choose our variables:

$$u(x) = x, \qquad v'(x) = \sin x$$

Now, we find $u'(x)$ and $v(x)$:

$$u'(x) = 1, \qquad v(x) = -\cos x$$

And now we replace in the formula:

$$\int \underbrace{x}_{u(x)} . \underbrace{\sin x}_{v'(x)} .dx = \underbrace{x}_{u(x)} .\underbrace{(-\cos x)}_{v(x)} - \int \underbrace{1}_{u'(x)} .\underbrace{(-\cos x)}_{v(x)} .dx$$

$$= -x \cos x + \underbrace{\int \cos x dx}_{\sin x} = -x \cos x + \sin x + C$$

$$\boxed{\int x \sin x dx = -x \cos x + \sin x + C}$$

Example 3

Find the integral:

$$\int \ln x dx$$

*Solution.* Integration by parts is very tricky by nature. In this example we are going to learn a little trick that allows us to use integration by parts even when we have only one function.

We write this integral as:

$$\int \ln x dx = \int \ln x.1.dx$$

We just introduced a factor 1 in the integral. Now, we choose our variables as:

$$u(x) = \ln x, \qquad v'(x) = 1$$

So, we have that:

$$u'(x) = \frac{1}{x}, \qquad v(x) = x$$

And applying integration by parts:

$$\int \underbrace{\ln x}_{u(x)} . \underbrace{1}_{v'(x)} .dx = \underbrace{\ln x}_{u(x)} . \underbrace{x}_{v(x)} - \int \underbrace{\frac{1}{x}}_{u'(x)} . \underbrace{x}_{v(x)} .dx$$

$$= x \ln x - \int \frac{1}{\cancel{x}} . \cancel{x} dx = x \ln x - \int dx = x \ln x - x + C$$

$$\boxed{\int \ln x dx = x(\ln x - 1) + C}$$

Example 4

Find the integral:

$$\int x^2 e^x dx$$

*Solution.* In this example we'll learn that sometimes we need to apply integration by parts more than once. Let's define:

$$u(x) = x^2, \qquad v'(x) = e^x$$

We have that:

$$u'(x) = 2x, \qquad v(x) = e^x$$

Applying integration by parts:

$$\int \underbrace{x^2}_{u(x)} . \underbrace{e^x}_{v'(x)} .dx = \underbrace{x^2}_{u(x)} . \underbrace{e^x}_{e^x} - \int \underbrace{2x}_{u'(x)} . \underbrace{e^x}_{v(x)} .dx$$

$$= x^2 e^x - 2 \int x e^x dx$$

Now, we use integration by parts again, this time on the integral that is left. Let's use the letters $w$ and $z$:

$$w(x) = x, \qquad z'(x) = e^x$$

$$w'(x) = 1, \qquad z(x) = e^x$$

Using integration by parts again:

$$x^2 e^x - 2 \int \underbrace{x}_{w(x)} . \underbrace{e^x}_{z'(x)} .dx = x^2 e^x - 2 \left( \underbrace{x}_{w(x)} . \underbrace{e^x}_{z(x)} - \int \underbrace{1}_{w'(x)} . \underbrace{e^x}_{z(x)} .dx \right)$$

$$= x^2 e^x - 2 \left( x e^x - \underbrace{\int e^x dx}_{e^x} \right) = x^2 e^x - 2x e^x + 2e^x + C$$

$$\boxed{\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C}$$

# 4 The LIATE Rule

Until now I haven't given any explanation of how I choose the functions $u(x)$ and $v'(x)$. The fact is that if you don't choose these correctly, your integral won't simplify and you may not be able to solve the integral.

The ***LIATE*** rule is a rule of thumb that helps us in choosing $u(x)$ and $v'(x)$ correctly. The rule specifies you what type of function you should choose as $u(x)$. And the rule is in the order of the letters in the word LIATE itself.

Each letter stands for a type of function:
L:**L**ogarithmic function
I: **I**nverse Trigonometric Function
A: **A**lgebraic Function
T: **T**rigonometric Function
E:**E**xponential Function

How do we use this rule? Whichever function comes first in the list should be $u(x)$. Let's see an example:

**Example 5**

Find the integral:

$$\int e^x \sin x dx$$

*Solution.* Here we have an exponential and a trigonometric function:

$$\int \underbrace{e^x}_{E} . \underbrace{\sin x}_{T} .dx$$

According to LIATE, the T comes before the E, so we should choose:

$$u(x) = \sin x, \qquad v'(x) = e^x$$

So, we have that:

$$u'(x) = \cos x, \qquad v(x) = e^x$$

And replacing in our integral:

$$\int \underbrace{e^x}_{v'(x)} . \underbrace{\sin x}_{u(x)} .dx = \underbrace{\sin x}_{u(x)} . \underbrace{e^x}_{v(x)} - \int \underbrace{\cos x}_{u'(x)} . \underbrace{e^x}_{v(x)} .dx$$

$$= e^x \sin x - \int e^x \cos x dx$$

We are going to apply integration by parts again to the integral that remains:

$$w(x) = \cos x, \qquad z'(x) = e^x$$

$$w'(x) = -\sin x, \qquad z(x) = e^x$$

Note that we used LIATE. Now:

$$\int e^x \sin x dx = e^x \sin x - \left[ \underbrace{\cos x}_{w(x)} . \underbrace{e^x}_{z(x)} - \int \underbrace{(-\sin x)}_{w'(x)} . \underbrace{e^x}_{z(x)} .dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

So, we have that:

$$\underbrace{\int e^x \sin x dx}_{!!!} = e^x \sin x - e^x \cos x + \underbrace{\int e^x \sin x dx}_{!!!}$$

The integral that we get at the right side of the equation is the same as the one in the left! We can add that integral to both sides to get:

$$2\int e^x \sin x dx = e^x \sin x - e^x \cos x$$

Now, we divide both sides by 2:

$$\boxed{\int e^x \sin x dx = \frac{e^x (\sin x - \cos x)}{2} + C}$$

# 5 Exercises

Find the integrals:

1. $\int x \ln x dx$
2. $\int x^2 \ln x dx$
3. $\int x \arcsin x dx$
4. $\int \frac{\arcsin x}{x^2} dx$

# 6 Still To Come

- **Day 18: Integrals by Trigonometric Substitution**
- **Day 19: Solving Trigonometric Integrals**
- **Day 20: Introduction to Definite Integrals**
- **Day 21: The Fundamental Theorem of Calculus**