Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Integrals

Day 18: Integrals by Trigonometric Substitution

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Contents

1 Welcome

Welcome to **Day 18** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we learn another technique of integration: **trigonometric substitution.**

2 An Example

We are going to learn another technique for calculating integrals. Let's see what is the idea with an example:

$$\int \frac{x^3}{\sqrt{1-x^2}} dx$$

We can't solve this integral with simple substitution, because we don't have x in the numerator, but x^3. We can observe that what is inside the square root looks pretty similar to a familiar trigonometric identity:

$$1 - \sin^2 t = \cos^2 t$$

So, we are tempted to make the substitution:

$$x = \sin t$$

We are introducing the variable t. Now, let's take the derivative of x with respect to t:

$$\frac{dx}{dt} = \cos t$$

Now, we've already seen that differentials behave like real numbers. So, we can *solve* for dx in the previous equation:

$$dx = \cos t \, dt$$

Now, let's replace these variables in our integral:

$$\int \frac{x^3}{\sqrt{1-x^2}} dx = \int \underbrace{\sin^3 t}_{x^3} . \underbrace{\frac{1}{\sqrt{1-\sin^2 t}}}_{\frac{1}{\sqrt{1-x^2}}} . \underbrace{\cos t \, dt}_{dx}$$

$$= \int \frac{\sin^3 t \cos t \, dt}{\sqrt{1-\sin^2 t}}$$

Now, we use the identity $\sqrt{1-\sin^2 t} = \cos t$ in the denominator:

$$\int \frac{\sin^3 t \cos t \, dt}{\sqrt{1-\sin^2 t}} = \int \frac{\sin^3 t \cancel{\cos t} dt}{\underbrace{\cancel{\cos t}}_{\sqrt{1-\sin^2 t}}} = \int \sin^3 t \, dt$$

We have now a trigonometric integral. One of the lessons of these course will focus on this type of integrals and how to solve them. For now, let's solve this particular case. We can write this integral as:

$$\int \sin^3 t\,dt = \int \sin^2 t \sin t\,dt$$

And again use the identity $\sin^2 t = 1 - \cos^2 t$:

$$\int \sin^2 t \sin t\,dt = \int \left(1 - \cos^2 t\right)\sin t\,dt$$

And break this into two integrals:

$$\int \left(1 - \cos^2 t\right)\sin t\,dt = \int \sin t\,dt - \int \cos^2 t \sin t\,dt$$

The first one of those is a direct integral:

$$\underbrace{\int \sin t\,dt}_{-\cos t} - \int \cos^2 t \sin t\,dt = -\cos t - \int \cos^2 t \sin t\,dt$$

And the integral that is left can be solved by substitution:

$$u = \cos t, \qquad \frac{du}{dt} = -\sin t$$

$$-\cos t - \int \cos^2 t \sin t\,dt = -\cos t - \int \underbrace{u^2}_{\cos^2 t} . \left(\underbrace{-\frac{du}{dt}}_{\sin t}\right) .dt$$

$$= -\cos t + \int u^2 . \frac{du}{\cancel{dt}} .\cancel{dt} = -\cos t + \int u^2 du = -\cos t + \frac{u^3}{3} + C$$

And we substitute back:

$$\int \sin^3 t\,dt = -\cos t + \frac{\cos^3 t}{3} + C$$

Now we need to substitute back our original variable x. We know that:

$$x = \sin t$$

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$$

So, we have that:

$$\int \frac{x^3}{\sqrt{1 - x^2}} dx = \underbrace{-\sqrt{1 - x^2}}_{-\cos t} + \frac{1}{3} . \left(\underbrace{\sqrt{1 - x^2}}_{\cos t}\right)^3 + C$$

$$= -\sqrt{1-x^2} + \frac{1-x^2}{3}.\sqrt{1-x^2} + C$$

So, our final answer is:

$$\boxed{\int \frac{x^3}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + \frac{1-x^2}{3}.\sqrt{1-x^2} + C}$$

3 Why Trigonometric Substitution Works

There are four cases in which a trigonometric substitution simplifies an integral.

3.1 First Case

First Case

When you have an expression of the form:

$$\sqrt{a^2 - x^2}$$

in your integral. Here a is a constant and x is the variable.

Solution. The first example we solved fits into this case. The substitution that works is:

$$\frac{x}{a} = \sin t$$

This works because if we do that:

$$\boxed{\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 t} = a\sqrt{1 - \sin^2 t} = a \cos t}$$

This transforms a complicated square root into a simple trigonometric function. This usually simplifies the integral.

3.2 Second Case

Second Case

When you have an expression of the form:

$$\sqrt{a^2 + x^2}$$

in your integral. Here, again, a is a constant and x is the variable.

Solution. In this case, we want to make use of the following trigonometric identity:

$$1 + \tan^2 t = \sec^2 t$$

To do that, we make the substitution:

$$\frac{x}{a} = \tan t$$

And we get:

$$\boxed{\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 t} = a\sqrt{1 + \tan^2 t} = a \sec t}$$

You can also use this substitution when you don't have the square root, that is, for expressions of the form:

$$a^2 + x^2$$

In that case, you will transform this into:

$$a^2 \sec^2 t$$

3.3 Third Case

Third Case

When you have an expression of the form:

$$\sqrt{x^2 - a^2}$$

in your integral.

Solution. Here we want to use again the same identity:

$$1 + \tan^2 t = \sec^2 t$$

But in the form:

$$\sec^2 t - 1 = \tan^2 t$$

So, we make the substitution:

$$\frac{x}{a} = \sec t$$

And we get:

$$\boxed{\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 t - a^2} = a\sqrt{\sec^2 t - 1} = a \tan t}$$

4 Another Example

Example 2

Let's find the integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

Solution. We have an expression of the form $\sqrt{a^2+x^2}$. So, we use the substitution:

$$x = \tan t$$

We have that:

$$\frac{dx}{dt} = \sec^2 t \Rightarrow dx = \sec^2 t dt$$

Now, making the substitution:

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{1}{\underbrace{\tan^2 t}_{x^2}} \cdot \frac{1}{\sqrt{1+\underbrace{\tan^2 t}_{x^2}}} \cdot \underbrace{\sec^2 t dt}_{dx}$$

$$= \int \frac{\sec^2 t dt}{\tan^2 t\sqrt{1+\tan^2 t}}$$

And we use the trig identity:

$$1+\tan^2 t = \sec^2 t$$

$$\int \frac{\sec^2 t dt}{\tan^2 t \underbrace{\sqrt{1+\tan^2 t}}_{\sec t}} = \int \frac{\sec^{\not 2} t dt}{\tan^2 t \cancel{\sec t}} = \int \frac{\sec t dt}{\tan^2 t}$$

$$= \int \frac{\frac{1}{\cos t}}{\frac{\sin^2 t}{\cos^2 t}} dt = \int \frac{\cos t dt}{\sin^2 t}$$

Now we have to solve this integral:

$$\int \frac{\cos t \, dt}{\sin^2 t}$$

We can do it by substitution:

$$u = \sin t, \qquad \frac{du}{dt} = \cos t$$

$$\int \frac{\cos t \, dt}{\sin^2 t} = \int \frac{du}{\cancel{dt}} \cdot \frac{\cancel{dt}}{u^2} = \int u^{-2} dt$$

$$= -\frac{1}{u} + C$$

And we substitute back:

$$\int \frac{\cos t \, dt}{\sin^2 t} = -\frac{1}{\sin t} + C$$

Our original substitution was:

$$x = \tan t$$

And using a little bit of trigonometry, we get that:

$$\tan t = \frac{\sin t}{\cos t} = \frac{\sin t}{\sqrt{1 - \sin^2 t}}$$

$$\tan^2 t = \frac{\sin^2 t}{1 - \sin^2 t}$$

$$\tan^2 t - \sin^2 t \tan^2 t = \sin^2 t$$

$$\sin^2 t \left(1 + \tan^2 t\right) = \tan^2 t$$

$$\sin t = \frac{\tan t}{\sqrt{1 + \tan^2 t}} = \frac{x}{\sqrt{1 + x^2}}$$

So, using this, we get the final answer:

$$\boxed{\int \frac{dx}{x^2 \sqrt{1 + x^2}} = -\frac{\sqrt{1 + x^2}}{x} + C}$$

5 Exercises

Solve the integrals by trigonometric substitution:

1. $\int \frac{\sqrt{a^2-x^2}}{x^2} dx$
2. $\int \frac{dx}{\sqrt{\left(a^2+x^2\right)^3}}$
3. $\int \frac{x^2-a^2}{x} dx$

6 Still To Come

- **Day 19: Solving Trigonometric Integrals**
- **Day 20: Introduction to Definite Integrals**
- **Day 21: The Fundamental Theorem of Calculus**