Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Integrals

Day 19: Trigonometric Integrals

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Contents

1 Welcome

Welcome to **Day 19** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we are going to learn how to solve **trigonometric integrals**.

2 Types of Trigonometric Integrals

Our goal is to learn how to solve **most** trigonometric integrals. A goal of knowing how to solve all trigonometric integrals is not realistic, because there are some seemingly simple integrals that do no have simple functions as primitives. For example:

$$\int \sin x^2 dx$$

We are going to learn how to solve any integral that is solvable by hand. And to do that, it will be useful to classify trigonometric integrals into four cases.

First Case: Integrals With Odd Powers of $\sin x$ or $\cos x$

For example:

$$\int \sin^3 x dx$$

$$\int \cos^4 x \sin^3 x dx$$

Second Case: Integrals With Only Even Powers of $\sin x$ or $\cos x$

For example:

$$\int \cos^2 x dx$$

Third case: Integrals Involving $\tan x$ or $\sec x$

For example:

$$\int \tan^3 x dx$$

Fourth: Just Tricky Trigonometric Integrals

For example:

$$\int \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx$$

This last four case is for integrals that do not fit into any of the other cases. Now, let's solve an example of each of these.

3 First Case: Odd Powers of $\sin x$ or $\cos x$

Example 1

Let's solve the integral:

$$\int \sin^3 x dx$$

Solution. We can write this integral as:

$$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$$

And use the identity:

$$\sin^2 x = 1 - \cos^2 x$$

$$\int \sin^2 x \sin x\, dx = \int \left(1 - \cos^2 x\right) \sin x\, dx$$

And we can break this up into two integrals:

$$\int \sin x\, dx - \int \cos^2 x \sin x\, dx$$

The first integral can be solved directly, and the second one by substitution:

$$u = \cos x, \qquad \frac{du}{dx} = -\sin x$$

$$\int \sin x\, dx - \int \cos^2 x \sin x\, dx =$$

$$= -\cos x - \int u^2 \cdot \left(-\frac{du}{dx}\right) . dx$$

$u^2 \to \cos^2 x$, $\left(-\frac{du}{dx}\right) \to \sin x$

That is:

$$-\cos x + \int u^2.\frac{du}{\cancel{dx}}.\cancel{dx} = -\cos x + \int u^2 du$$

$$-\cos x + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C$$

$$\boxed{\int \sin^3 x\, dx = -\cos x + \frac{\cos^3 x}{3} + C}$$

4 Second Case: Even Powers of $\sin x$ or $\cos x$

Example 2

Let's solve the integral:

$$\int \cos^2 x \, dx$$

Solution. When we have even powers of $\sin x$ or $\cos x$, the trick is to use the following trig identity:

$$\cos 2x = \cos(x + x) = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

That is:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Using the first of these identities in our integral:

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx$$

And we can now solve this integral by substitution:

$$u = 2x, \qquad \frac{du}{dx} = 2$$

$$\int \frac{1 + \cos 2x}{2} dx = \int \frac{1 + \cos u}{2} . \underbrace{\left(\frac{1}{2} . \frac{du}{dx}\right)}_{1} . dx$$

And we get:

$$\int \frac{1+\cos u}{4} . \frac{du}{\cancel{dx}} . \cancel{dx} = \int \frac{1+\cos u}{4} du$$

And this integral is solved easily:

$$\int \frac{1+\cos u}{4} du = \frac{1}{4} \int du + \frac{1}{4} \int \cos u du$$

$$= \frac{u}{4} - \frac{\sin u}{4} + C$$

And now we replace back $u = 2x$:

$$\boxed{\int \cos^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C}$$

5 Third Case: Integrals Involving $\tan x$ or $\sec x$

Example 3

Let's solve the integral:

$$\int \tan^3 x \, dx$$

Solution. We want to use the identity:

$$1 + \tan^2 x = \sec^2 x$$

And the fact that:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

So, we write this integral as:

$$\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx$$

And replace $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \tan x \, dx = \int \underbrace{\left(\sec^2 - 1\right)}_{\tan^2 x} \tan x \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \frac{\sin x}{\cos x} dx$$

And we can solve both of these integrals by substitution:

$$u = \tan x, \qquad \frac{du}{dx} = \sec^2 x$$

$$v = \cos x, \qquad \frac{dv}{dx} = -\sin x$$

$$\int \tan x \sec^2 x \, dx - \int \frac{\sin x}{\cos x} dx$$

$$= \int \underbrace{u}_{\tan x} . \underbrace{\frac{du}{dx}}_{\sec^2 x} . dx - \int \underbrace{\frac{1}{v}}_{\frac{1}{\cos x}} . \underbrace{\left(-\frac{dv}{dx}\right)}_{\sin x} . dx$$

$$\int u . \frac{du}{\cancel{dx}} . \cancel{dx} + \int \frac{1}{v} . \frac{dv}{\cancel{dx}} . \cancel{dx}$$

$$\int u \, du + \int \frac{dv}{v} = \frac{u^2}{2} + \ln v + C$$

And we substitute back $u = \tan x$ and $v = \cos x$:

$$\boxed{\int \tan^3 x \, dx = \frac{\tan^2 x}{2} + \ln \cos x + C}$$

6 Fourth Case: Tricky Trigonometric Integrals

Example 3

Let's solve the integral:

$$\int \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx$$

Solution. In this case we can only guess what trigonometric identity will do the trick. So, first of all, we recognize that the denominator looks somewhat like a perfect square. For example, we have that:

$$\left(\cos^2 x + \sin^2 x\right)^2 = \cos^4 x + 2\sin^2 x \cos^2 x + \sin^4 x = 1$$

So:

$$\cos^4 x + \sin^4 x = 1 - 2\sin^2 x \cos^2 x = 1 - \frac{\sin^2 2x}{2}$$

Here we used the identities:

$$\cos^2 x + \sin^2 x = 1$$

And:

$$\sin 2x = 2\sin x \cos x \Rightarrow \frac{\sin^2 2x}{2} = 2\sin^2 x \cos^2 x$$

Replacing this in our integral:

$$\int \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx = \int \frac{\sin 2x}{1 - \frac{\sin^2 2x}{2}}\, dx = \int \frac{2\sin 2x}{2 - \sin^2 2x} dx$$

Now, this looks a little more familiar. We can try the substitution:

$$u = 2x, \qquad \frac{du}{dx} = 2$$

$$\int \frac{2\sin 2x}{2 - \sin 2x}\,dx = 2\int \frac{\sin u}{2 - \sin^2 u} \cdot \underbrace{\left(\frac{1}{2}.\frac{du}{dx}\right)}_{1} .dx$$

So, we must solve the integral:

$$\cancel{2}\int \frac{\sin u}{2 - \sin u}.\frac{1}{\cancel{2}}.\frac{du}{\cancel{dx}}.\cancel{dx} = \int \frac{\sin u}{2 - \sin^2 u}.du$$

Now, what comes to mind, is using $\sin^2 u = 1 - \cos^2 u$:

$$\int \frac{\sin u}{2 - \sin^2 u}.du = \int \frac{\sin u}{2 - 1 + \cos^2 u}du = \int \frac{\sin u}{1 + \cos^2 u}du$$

And now we try the substitution:

$$v = \cos u, \qquad \frac{dv}{du} = -\sin u$$

$$\int \frac{\sin u}{1 + \cos^2 u}du = \int \frac{1}{1 + v^2}.\left(-\frac{dv}{\cancel{du}}\right).\cancel{du}$$

$$= -\int \frac{dv}{1 + v^2}$$

And this integral is solved directly, since $(\arctan v)' = \frac{1}{1+v^2}$:

$$-\int \frac{dv}{1 + v^2} = -\arctan v + C$$

And now, we substitute back $v = \cos u = \cos 2x$:

$$\boxed{\int \frac{\sin 2x}{\cos^4 x + \sin^4 x}\,dx = -\arctan\cos 2x + C}$$

7 Exercises

Solve the integrals:

1. $\int \cos^3 x dx$

2. $\int \sin^2 x dx$

3. $\int \sin^4 x dx$

4. $\int \cos^4 x \sin^3 x \, dx$

5. $\int \frac{dx}{(1+\cos x)^2}$ (this one is tricky!)

8 Still To Come

- **Day 20: Introduction to Definite Integrals**
- **Day 21: The Fundamental Theorem of Calculus**