Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Integrals

Day 20: Introduction to Definite Integrals

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Contents

1 Welcome

Welcome to **Day 20** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we are going to introduce the concept of **definite integral.**

2 Physical Intuition

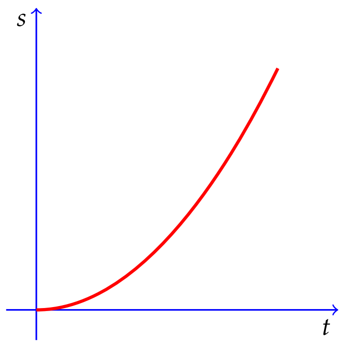
We are going to introduce the definite integral as the area under a curve. However, we're going to see first the physical interpretation of this concept. As with the derivative, we can see the integral either geometrically or physically.

2.1 The Solution to an Interesting Problem

Let's say that we have a moving object, such that its position is given by the following formula:

$$s(t) = \frac{t^2}{2}$$

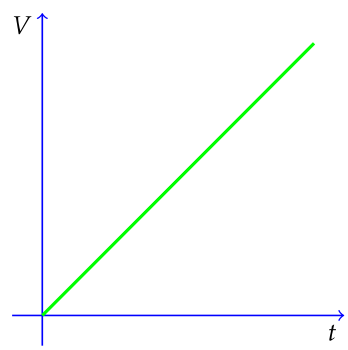
Here t represents time. So, this formula gives us the position of the object as it changes over time. Let's graph this function:



If we take the derivative of this function, we get another function of time:

$$s'(t) = t$$

We learned when we studied derivatives that this derivative can be viewed as velocity. That's the physical interpretation of the derivative. The graph of this specific function is a straight line:



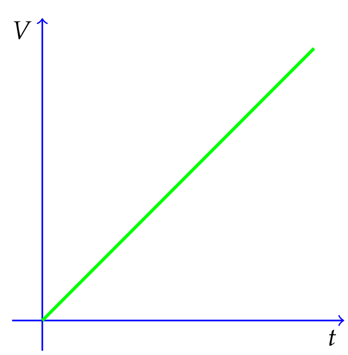
Now, let's consider the following problem. Let's say that we are given the velocity of the object at each instant of time. That is, we have as data $s'(t)$ for each t. Can we reconstruct $s(t)$? That is, can we find **the position of the object at any given time?**

This is an interesting question. And trying to solve it, mathematicians came to the concept of integral.

As a first step, let's try to simplify the problem: what if velocity were constant? If velocity is constant, we know from physics that distance is given by velocity times time. So, we can take any interval of time, multiply it by velocity, and we obtain the distance traveled by the object. This case is easy.

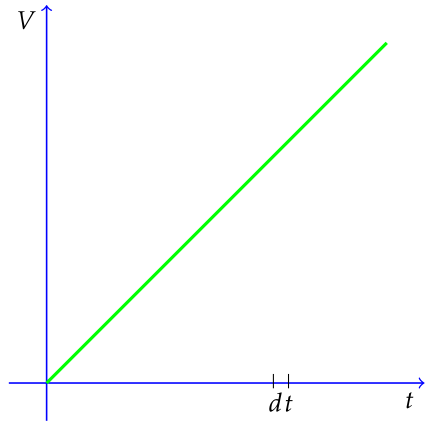
Let's now consider the particular case of the function we just saw:

$$V(t) = s'(t) = t$$

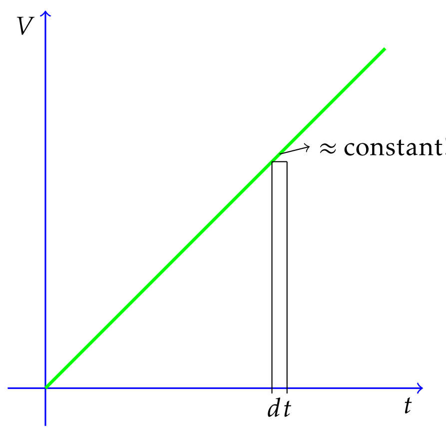


This function gives us the velocity of the object at each instant of time. How can we deduce the distance it has traveled? And the idea, as in most problems in mathematics, is to reduce the problem to the simpler case.

And to do that, we take a small, very small, time interval:



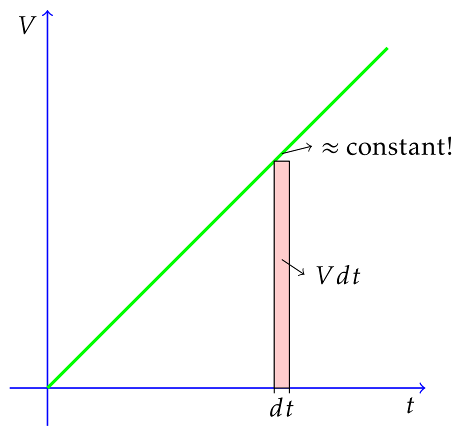
If we look closelly at the values of the function in this interval, we see that they do not change much. That is, the function is *almost* constant in this interval:



We can improve this approximation by taking the interval even smaller. Now, considering that velocity is constant in this time interval, we can calculate the distance traveled during that time:

$$ds = V\,dt$$

Because distance is velocity times time. Here we use the symbol ds to denote distance because this distance should be very small. Now, physically, we know that ds is distance, but what does it represent on the graph?

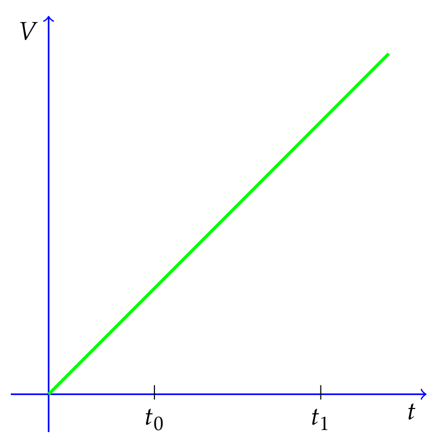


It is the area of the rectangle in the picture! Here we start to see the connection between this physical problem and the geometric concept of area.

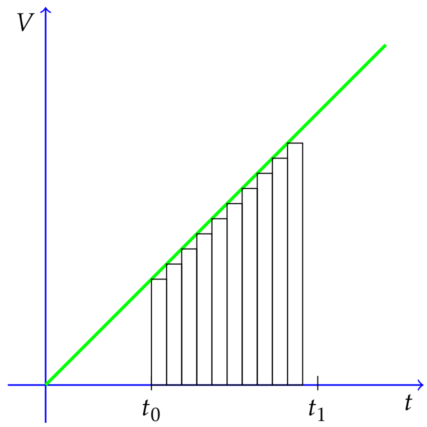
2.2 Considering Longer Intervals

We solved the problem in the case of very, very small time intervals. That is, given the function velocity, we can find the distance traveled during a very small time interval. Now, how do we find the distance traveled in longer interval?

Let's visualize the problem, we want to find the distance traveled in an interval $[t_0, t_1]$:



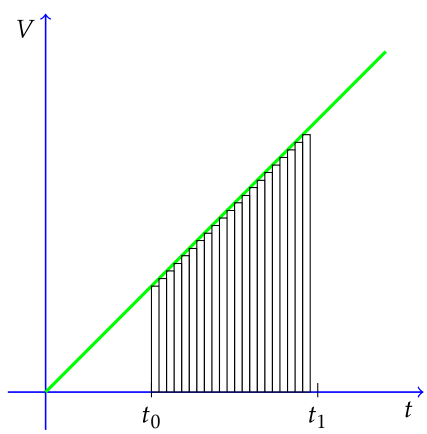
We cannot consider that velocity is constant in this larger interval, because that would be a very bad approximation. However, we can break up the interval into many small subintervals:



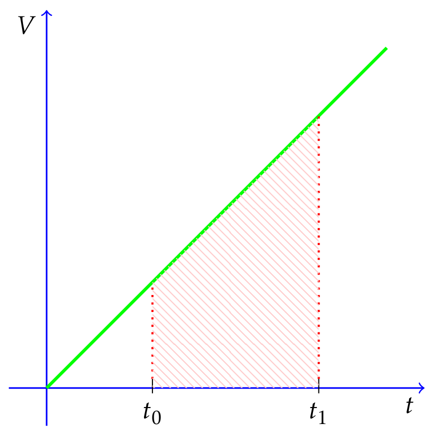
We can consider each of these rectangles of width dt. We know how to calculate the distance traveled in each of these smaller rectangles. So, to obtain the distance traveled in the bigger interval, we sum the distances traveled in all the smaller intervals!

Remember that this is only an approximation, because we need to take dt infinitely small to make the assumption that V is constant valid. So, guess what we are going to do now? We take the limit as the width dt approaches zero!

By taking smaller rectangles, we get the following figure:



And you can guess where this would lead. Taking the limit as dt approaches zero, the sum of the areas becomes the **area under the curve:**



And this is how we'll define the definite integral.

2.3 Definition

We are going to define the definite integral as the limit of the sums of the areas of the small rectangles, as the width dt of all rectangles approach 0. And as we are taking an infinite sum, we use a symbol that looks like an S for integrals:

$$\int$$

And the area of each small rectangle is given by:

$$V dt$$

Just remember that the base is dt and the height is V. However, V is any function, so we are going to use $f(x)$ instead of V. And we are going to use x as independent variable. Having this in mind, we denote the definite integral of $f(x)$, from x_0 to x_1 as:

$$\int_{x_0}^{x_1} f(x)dx$$

Now you can understand a few things you probably have been asking about, like where the symbol dx comes from. It is just the width of a very, very small rectangle.

The numbers x_0 and x_1 that appear in the symbol for integral are called the *lower limit* and *upper limit*, respectively. They only indicate the interval on which you are integrating.

3 Some Properties

Let's talk now about two simple properties of definite integrals. Most of these are properties shared with indefinite integrals. We mention them here so you can use them when you start solving definite integrals using the fundamental theorem of calculus.

Property 1: Take Constants Out

You can take constants out of the integral sign. For example:

$$\int_0^1 3x^2\,dx = 3\int_0^1 x^2\,dx$$

Property 2: Property of Sums

As with indefinite integrals, the integral of the sum of two functions equals the sum of the integrals of each function. For example:

$$\int_1^3 (x + \sin x)\,dx = \int_1^3 x\,dx + \int_1^3 \sin x\,dx$$

There are some other important properties of integrals, like the *Mean Value Theorem for Integrals*. This property can be proved as a theorem, but we won't do that here. You can view the intuitive idea in the presentation for Day 20.

We still don't know how to calculate definite integrals. That's why we won't have exercises today. However, we'll learn how to do that in the next lesson, with the **fundamental theorem of calculus.**

4 Still To Come

- **Day 21: The Fundamental Theorem of Calculus**