Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Integrals

Day 21: The Fundamental Theorem of Calculus

By Pablo Antuna

Contents

1 Welcome

Welcome to **Day 21** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus! You can find the rest of the lessons (which I call "*days*"), on this web page: Free Online Calculus Course.

Today we are going to focus on the **Fundamental Theorem of Calculus.**

2 The Area Function

Let's consider a function $f(t)$. Here we are using the letter t as independent variable, and you'll see why in a moment. Let's graph this function:

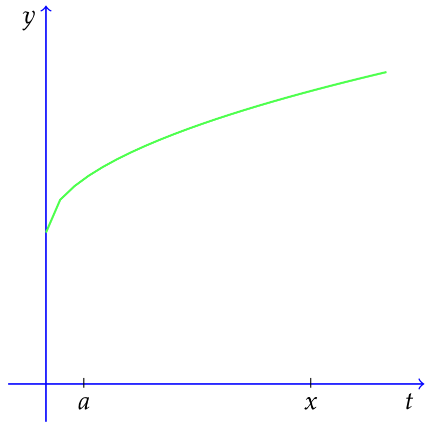


Figure 2.1: The function we are going to use to explain the *area function*.

Let's consider also two points a and x on the t axis. We are going to define another function, $A(x)$, that will give us the area under the graph from a to x:

$$A(x) = \int_a^x f(t)\,dt$$

This is just a normal function that we are defining. This definition is totally valid. Geometrically, what $A(x)$ represents is the area under the curve from a to x:

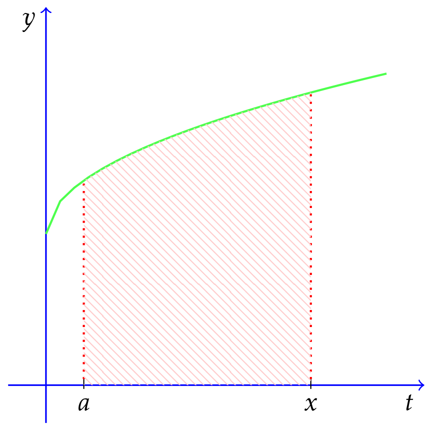
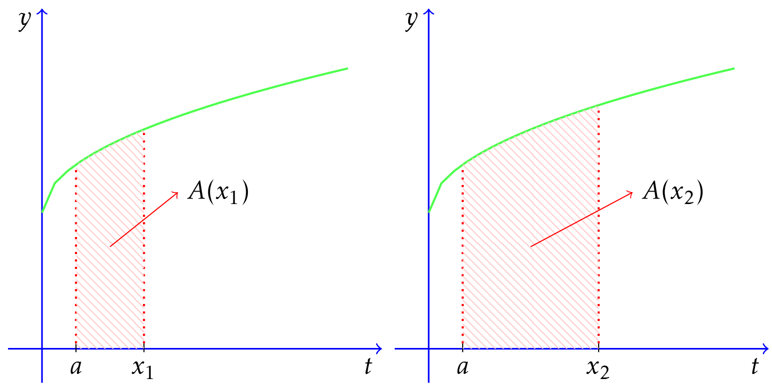


Figure 2.2: The colored area represents the value $A(x)$.

That's because the definition of definite integral. To get you used to this *area function*, let's see what other values of the function represent:



(a) The colored area represents $A(x_1)$. **(b)** The colored area represents $A(x_2)$.

Figure 2.3: Two different values of $A(x)$.

3 Intuitive Idea

Now that we introduced the area function, we are ready to enunciate the **Fundamental Theorem of Calculus.** This theorem simply says that:

$$A'(x) = f(x)$$

Simply that. Now, let's try to understand this short statement.

What this theorem is saying is that the rate of change of the area function equals $f(x)$. And geometrically, $f(x)$ is the following height:

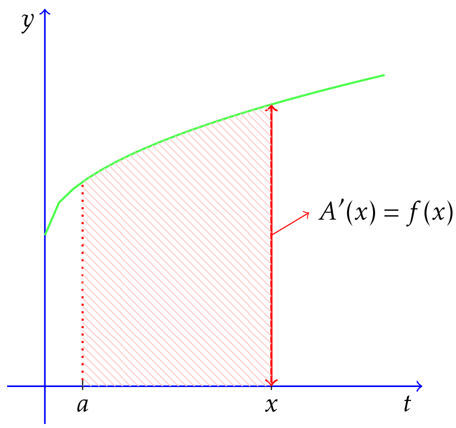


Figure 3.1: This is the geometrical interpretation of the Fundamental Theorem of Calculus.

I'm going to try to show you an *intuitive proof*. If you understand this intuitive proof, the formal proof will be much easier to understand. We begin with the definition of the derivative:

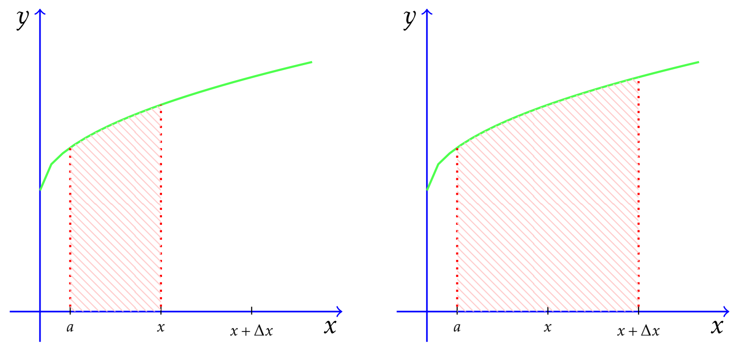
$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}$$

Let's try to give a geometrical meaning to the numerator $A(x+\Delta x) - A(x)$. We are going to take a positive Δx (it could also be negative). We have that:

$$A(x+\Delta x) = \int_a^{x+\Delta x} f(t)\,dt$$

$$A(x) = \int_a^{x} f(t)\,dt$$

So, we can represent these two numbers as the colored areas in the following graphs:



(a) The colored area represents $A(x)$. **(b)** This area represents $A(x+\Delta x)$.

Figure 3.2: We can see the meaning of $A(x)$ and $A(x+\Delta x)$ in these graphs.

We want to know what is the difference $A(x+\Delta x) - A(x)$. And this is a difference of areas. When you take $A(x)$ out of $A(x+\Delta x)$, what is left is the following area:

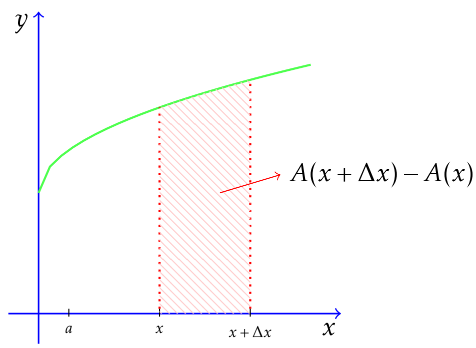


Figure 3.3: This area is $A(x+\Delta x) - A(x)$.

That is the area under the curve from x to $x+\Delta x$. So, now, as we want to find the derivative, we must make $\Delta x \to 0$. What will happen with this area when we do this? When Δx becomes small, we have something similar to this:

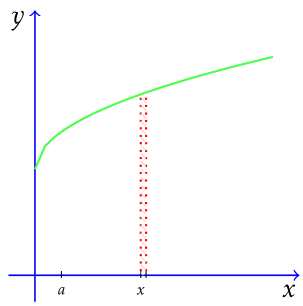


Figure 3.4: As $\Delta x \to 0$, $A(x+\Delta x) - A(x)$ becomes smaller. And looks like a rectangle...

But not only does $A(x+\Delta x) - A(x)$ become smaller, it also looks like a small rectangle. If we zoom, the upper part of this area looks like this:

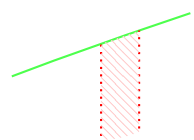


Figure 3.5: A close up view of the graph above.

We can approximate this area as a rectangle, because the error we would make is very small. That is, we can make the approximation:

$$A(x+\Delta x) - A(x) \approx f(x)\Delta x$$

The right side of this equation is the area of a rectangle of base Δx and height $f(x)$. So, if we introduce this approximation in our calculation of the derivative, we get:

$$\frac{A(x+\Delta x) - A(x)}{\Delta x} \longrightarrow \frac{f(x)\Delta x}{\Delta x} = \frac{f(x)\cancel{\Delta x}}{\cancel{\Delta x}} = f(x)$$

That is:

$$A'(x) = f(x)$$

And this result is pretty intuitive. The rate of change of the area under the curve equals the height of the curve at the extreme point. If the height is greater, the rate of change will be greater, the area will increase more. And viceversa.

If this little intuitive argument that I presented does not convince you completely, you can read in the next section a complete proof using the Mean Value Theorem for integrals.

4 Formal Proof

For the formal proof, we need to assume that the function $f(t)$ is continuous. Given this function, we define the area function:

$$A(x) = \int_a^x f(t)\,dt$$

And we wish to calculate its derivative:

$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}.\left[A(x+\Delta x) - A(x)\right]$$

To begin with, we introduce the definition of $A(x)$ into the definition of derivative:

$$A'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}.\left[\underbrace{\int_a^{x+\Delta x} f(t)\,dt}_{A(x+\Delta x)} - \underbrace{\int_a^x f(t)\,dt}_{A(x)}\right]$$

We can separate the first integral into two integrals like this:

$$\int_a^{x+\Delta x} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+\Delta x} f(t)\,dt$$

That is:

$$\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt$$

And we can replace this in the limit we are trying to calculate:

$$A'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}.\left[\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\,dt$$

Here is when we need to apply the **Mean Value Theorem for integrals.** This theorem says that, given a function $f(t)$ defined in the interval $[a,b]$, then there is a point c in the interval $[a,b]$ such that:

$$\int_a^b f(t)\,dt = f(c)(b-a)$$

Applying this theorem to the integral that we have in our limit, we get:

$$\int_x^{x+\Delta x} f(t)\,dt = f(c)\left[(x+\Delta x)-x\right] = f(c)\Delta x$$

Now, we can do this for each Δx. That is, what we really have is:

$$\int_x^{x+\Delta x} f(t)\,dt = f\left[c(\Delta x)\right]\Delta x$$

Here $c(\Delta x)$ is a function of Δx. This function is such that:

$$x \le c(\Delta x) \le x+\Delta x$$

This function $c(\Delta x)$ is given by the Mean Value Theorem for integrals, because the theorem says that there is a c for each Δx. That is, we can define such a function. Now we need to calculate the limit:

$$A'(x) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}.f\left[c(\Delta x)\right].\Delta x = \frac{1}{\cancel{\Delta x}}.f\left[c(\Delta x)\right].\cancel{\Delta x} = \lim_{\Delta x\to 0} f\left[c(\Delta x)\right]$$

And now, we can solve this using the **Squeeze Theorem** and the fact that $f(t)$ is continuous. From the inequality:

$$\underset{\displaystyle \swarrow \atop \displaystyle x}{x} \le c(\Delta x) \le \underset{\displaystyle \searrow \atop \displaystyle x}{x+\Delta x}$$

we deduce, using the **Squeeze Theorem**, that:

$$\lim_{\Delta x\to 0} c(\Delta x) = x$$

That's because boths the left side and the right side of the inequality approach x when $\Delta x \to 0$. And now, using the fact that $f(t)$ is continuous, we have that:

$$\lim_{\Delta x\to 0} f\left[c(\Delta x)\right] = f\left[\lim_{\Delta x\to 0} c(\Delta x)\right] = f(x)$$

That is:

$$\boxed{A'(x) = f(x)}$$

And that's what we wanted to prove.

5 Newton-Leibniz Formula

Until now, the Fundamental Theorem of Calculus might seem only a mathematical curiosity. But now, we are going to see why it is really fundamental. We are going to show the **Newton-Leibniz Formula**, which gives us a closed formula for calculating definite integrals.

We proved that the area function:

$$A(x) = \int_a^x f(t)\,dt$$

is such that:

$$A'(x) = f(x)$$

In other words, the function $A(x)$ is a *primitive* of $f(x)$. When we learned about indefinite integrals, we proved that the difference between any two primitives is a constant. Let's say that $F(x)$ is another primitive of $f(x)$. Then:

$$A(x) - F(x) = \int_a^x f(t)\,dt - F(x) = C$$

where C is a constant. Now, we know this equation is valid for any x. So, let's choose $x = a$:

$$\int_a^a f(t)\,dt = F(a) + C$$

But, what is this integral:

$$\int_a^a f(t)\,dt = ?$$

We can deduce its value from the following graph:

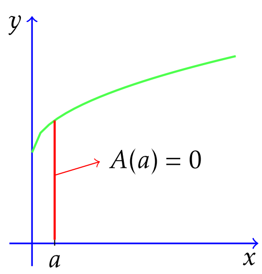


Figure 5.1: $A(a)$ is the *area* of the vertical line.

The value of this integral is zero! So:

$$\int_a^a f(t)\,dt = F(a) + C = 0$$

$$\boxed{C = -F(a)}$$

Now, let's choose $x = b$ in the original formula:

$$\int_a^b f(t)\,dt = F(b) + C$$

But we know that $C = -F(a)$, so:

$$\boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

What we've found is an efficient method for calculating definite integrals. The method is:

1. Find a primitive $F(x)$ of the function $f(x)$ (we're already experts at this).

2. Apply the formula we just proved.

This means that the calculation of a definite integral is reduced to the calculation of an indefinite integral. This result is really, really useful. In the next section we are going to see some examples of this.

6 Examples

Example 1

Let's find the definite integral:

$$\int_0^1 x^2\,dx$$

Solution. We are going to apply the Fundamental Theorem of Calculus. To do that, we need to find the indefinite integral first:

$$\int x^2\,dx = \frac{x^3}{3} + C$$

So, we are going to take the primitive:

$$F(x) = \frac{x^3}{3}$$

And using the Newton-Leibniz formula:

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3}$$

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$

And this result is pretty amazing. The area we just found is the following:

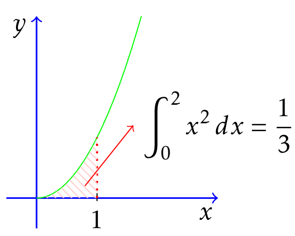


Figure 6.1: The area found in Example 1.

Example 2

Let's find the definite integral:

$$\int_0^{\pi} \sin x \, dx$$

Solution. As in the first example, we first find the indefinite integral:

$$\int \sin x \, dx = -\cos x + C$$

And we consider the antiderivative:

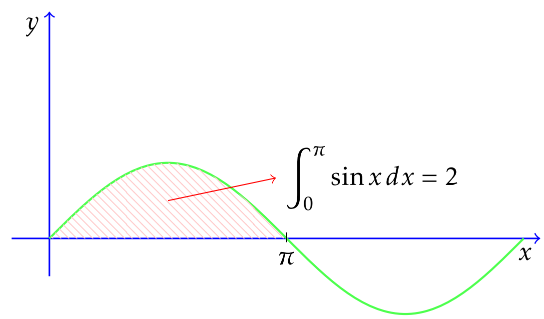
$$F(x) = -\cos x$$

And now we apply the Fundamental Theorem of Calculus:

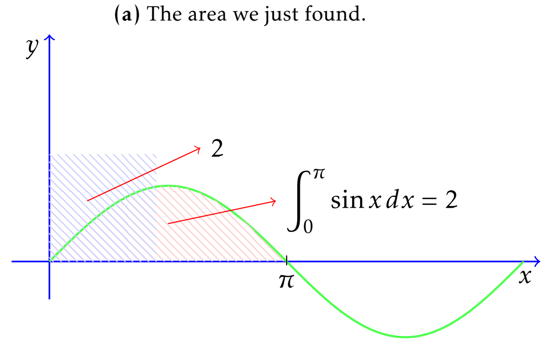
$$\int_0^{\pi} \sin x \, dx = F(\pi) - F(0) = -\underbrace{\cos \pi}_{-1} - \left(-\underbrace{\cos 0}_{1} \right) = -(-1) - (-1)$$

$$\boxed{\int_0^{\pi} \sin x \, dx = 2}$$

This result is also pretty amazing. We can see the area we just found in the following graphs:



(a) The area we just found.



(b) A little box of the same area.

Figure 6.2: The area we just found, and the same area in a rectangular form for comparison.

7 Exercises

Solve the following integrals using the fundamental theorem of calculus:

1. $\int_0^1 e^x \, dx$
2. $\int_0^1 \frac{dx}{1+x^2}$
3. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$
4. $\int_1^3 \frac{dx}{2x-1}$

If you want to see the answers, please go to the following web-page: Answers.