Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Limits

Day 4: Limits by Rationalization

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1 Welcome

Welcome to Day 4 of the **Free Intuitive Calculus Course!** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on another useful technique for solving limits: rationalization.

2 The Principle of Rationalization

To learn this technique, first of all we need to establish one important property of square roots, since we'll be dealing with them here. That is, that we can simply plug the value that x is approaching into a limit that involves square roots.

In symbols:

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{f(a)}$$

This is true if the function f is continuos. Since all the functions we'll be dealing with are continuous, we don't need to worry much about that requirement. What we DO need to worry about is whether or not the expression makes sense.

For example, if $f(a)$ is negative, the expression doesn't make sense, because there is no (real) square root of a negative number.

Now we're ready to learn the basic technique for solving limits involving square roots: rationalization.

3 A First Example

Let's find the limit:

$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}$$

Here if we simply replace h by 0 we get 0/0. So, we must look for another strategy. If you look back at your algebra days, you might remember of something called rationalization.

In rationalization, we try to kill the roots in the numerator. To accomplish this, we multiply by a convenient expression of 1. The expression inside the limit sign isn't altered because anything multiplied by 1 is the same anything.

Now, in this case, what makes the trick is what is called the conjugate. The conjugate of an expression $a+b$ is $a-b$. That is, we change the sign in the middle. So, the convenient expression of 1 is:

$$\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}=1$$

Multiplying the function in our limit we get:

$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}$$

Now we note that in the numerator we have a perfect square (or you can perform the multiplication there):

$$\lim_{h\to 0}\frac{(\sqrt{a+h})^2-(\sqrt{a})^2}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{h}{h(\sqrt{a+h}+\sqrt{a})}$$

Now we can simplify the h (h is never zero):

$$\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}$$

And now we can finally use the property we first talked about in this section: the continuity of the square root. We can simply replace h by 0:

$$\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\sqrt{a+0}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

4 A Second Example

Let's solve now the limit:

$$\lim_{x\to 0}\frac{\sqrt{a+x^2}-\sqrt{a-x^2}}{x^2}$$

If we plug x into the expression we get $0/0$. We need to apply rationalization. In this case, the conjugate of the numerator is $\sqrt{a+x^2}+\sqrt{a-x^2}$. So, we multiply both the numerator and denominator by that quantity:

$$\lim_{x\to 0}\frac{\left(\sqrt{a+x^2}-\sqrt{a-x^2}\right)\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}$$

The numerator is a perfect square and becomes:

$$\lim_{x\to 0}\frac{a+x^2-(a-x^2)}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}=$$

$$\lim_{x\to 0}\frac{2x^2}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}$$

Now the x^2 simplifies and we are left with:

$$\lim_{x\to 0}\frac{2}{\sqrt{a+x^2}+\sqrt{a-x^2}}$$

We can now simply plug the 0 into the expression and get:

$$\lim_{x\to 0}\frac{2}{\sqrt{a+x^2}+\sqrt{a-x^2}}=\frac{2}{\sqrt{a}+\sqrt{a}}=\frac{1}{\sqrt{a}}$$

5 Exercises

Solve the following limits by rationalization:

1. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$
2. $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$
3. $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$

6 Still To Come

- **Day 5: Squeeze Theorem**
- **Day 6: Trigonometric Limits**
- **Day 7: Limits at Infinity**