Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Limits

Day 5: The Squeeze Theorem

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1 Welcome

Welcome to **Day 5** of the **Intuitive Online Calculus Course!** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on an important theorem that will prove useful for finding many limits, specially trigonometric ones.

2 Intuitive Idea

The squeeze theorem is a very simple, commonsensical and even obvious idea. It may seem complicated in your book, but it's not. Here you will learn the intuition behind it and how to apply it to solve limits.

First, let's form an intuition of what it is about. Let's consider the following statements:

- Pablo always gets a better grade than Peter's or the same.
- Pablo always gets a worse grade than Mary's or the same.

This is, on any given exam, Pablo always gets a better grade than Peter's (or the same). And also, the same grade is worse than Mary's (or the same).

Now, let's consider the following case. On wednesday:

- Peter got a B
- Mary got a B

What is Pablo's grade? It is also B, right? You could say that Pablo has been squeezed between Peter and Mary.

This is what the squeeze theorem says. Really simple!

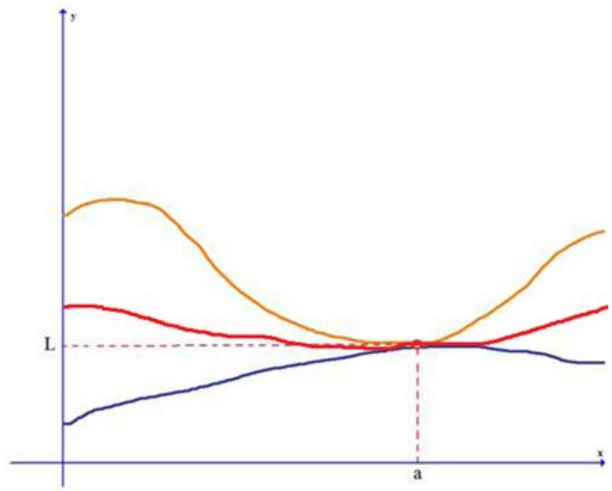
Let's try now to translate this to the language of calculus. Let's say f and g are two functions such that one is always below the other. Let's say that g is always above:

$$f(x) \leq g(x)$$

Also, let's say that their limits are equal in some point:

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$$

Now, let's suppose we squeeze a third function between them:



Let's call this third function h. So, we have that:

$$f(x) \leq h(x) \leq g(x)$$

In our graph we can see that as $x \to a$, $h(x)$ also approaches L, as the other two functions. That is, it has been squeezed by the two functions f and h. What we mean is that:

$$\lim_{x \to a} h(x) = L$$

So, what the squeeze theorem says is the following:

The Squeeze Theorem: Given three functions f, g and h; such that $f(x) \leq h(x) \leq g(x)$, if:

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$$

Then, the limit of h is the same, that is:

$$\lim_{x \to a} h(x) = L$$

.

This is a very intuitive idea, so I expect that I convinced you of its plausibility. The formal proof is also very simple. If you are interested in reading it, you must first learn the ε-δ definition of limits, a subject you can read about here: The Formal Limit Definition.

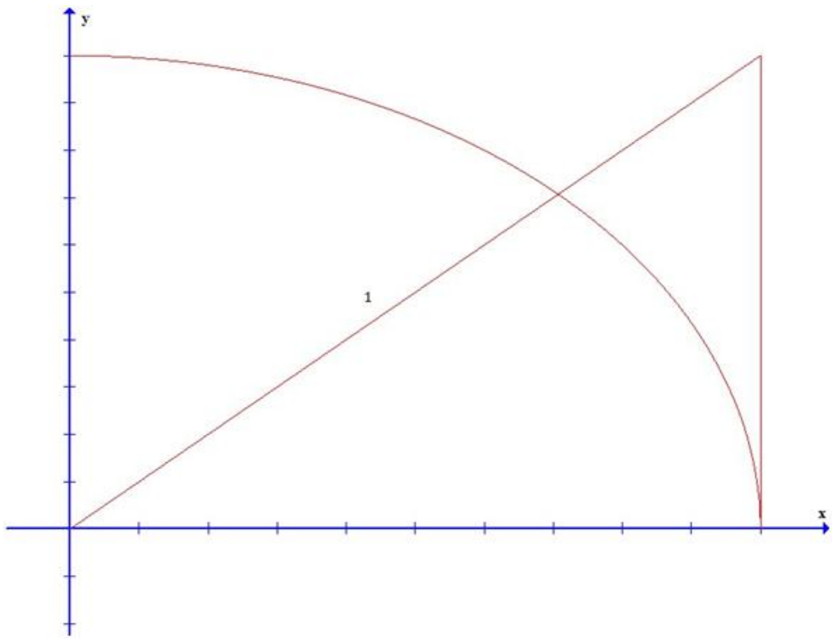
The meaning and usefulness of this theorem may become clearer in the following important application.

3 The Limit of $\frac{\sin x}{x}$

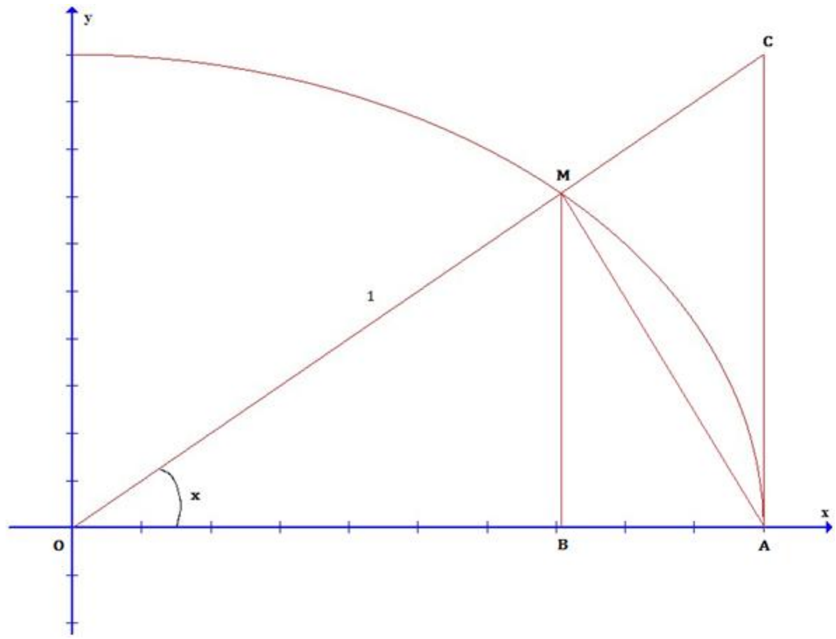
In this section we will prove the following limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

We will do it using geometry and the squeeze theorem. First of all let's graph a circle of radius 1 and a triangle like this:



Let's say x is the angle in the triangle, and let's also put names to the points. Let's also draw two extra lines:



We will use the geometric relations here to solve the limit. We will try to form an inequality in order to apply the squeeze theorem.

First of all, let's note that the area of triangle MOA is smaller than the area of the corresponding circular sector MOA. The circular sector is the slice of pizza that contains triangle MOA. Here we have our first inequality.

Next, let's note that the area of sector MOA is smaller than the area of triangle COA. So, we have:

$$\Delta MOA < \text{Sector } MOA < \Delta COA$$

Now, we only have to replace the actual values of these areas to have our inequality. Let's first calculate the area of triangle MOA. The area is base times height over 2. Let's note that:

$$\sin x = \frac{\overline{BM}}{1} = \overline{BM}$$

So, our area is:

$$\Delta MOA = \frac{1.\overline{BM}}{2} = \frac{\sin x}{2}$$

We obtain the area of triangle COA in a similar manner, but using the tangent function:

$$\tan x = \frac{\overline{AC}}{1} = \overline{AC}$$

And the area is:

$$\Delta COA = \frac{1.\overline{AC}}{2} = \frac{\tan x}{2}$$

Now we get to the area of the circular sector. You might remember (or not) that the area of a circular sector of radius r and angle θ is:

$$A = \frac{\theta r^2}{2}$$

A simple derivation of this formula is the following. The area of a circle is πr^2. A circle is a circular sector of angle $\theta = 2\pi$ (in radians). So, we have:

$$A(2\pi) = \pi r^2$$

If we multiply both sides of this equation by $\frac{\theta}{2\pi}$:

$$\frac{\theta}{2\pi} A(2\pi) = A(\frac{\theta 2\pi}{2\pi}) = A(\theta) = \frac{\theta \pi r^2}{2\pi} = \frac{\theta r^2}{2}$$

Now, we have a circle of radius 1 and angle x, so, our area is:

$$\text{Sector } MOA = \frac{x.1^2}{2} = \frac{x}{2}$$

If we plug these values into our inequality we get:

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{x}$$

Now we need to remember some things about the algebra of inequalities. We can multiply everything by 2 and the inequality holds:

$$\sin x < x < \tan x$$

We can replace $\tan x = \frac{\sin x}{\cos x}$:

$$\sin x < x < \frac{\sin x}{\cos x}$$

And divide everything by $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Let's remind ourselves that this is valid only when $\sin x$ is positive. In this case we are considering an angle x less than $90°$, so what we did is valid. If $\sin x$ were negative, dividing by it would had reversed the inequality.

Now, watching each side of the inequality, we can replace each member by its inverse and the equality will be inverted:

$$1 > \frac{\sin x}{x} > \cos x$$

What we did is to "solve" the inequality for $\frac{\sin x}{x}$. Here we are ready to apply the squeeze theorem. The first condition of the theorem is met. Now, we also know that:

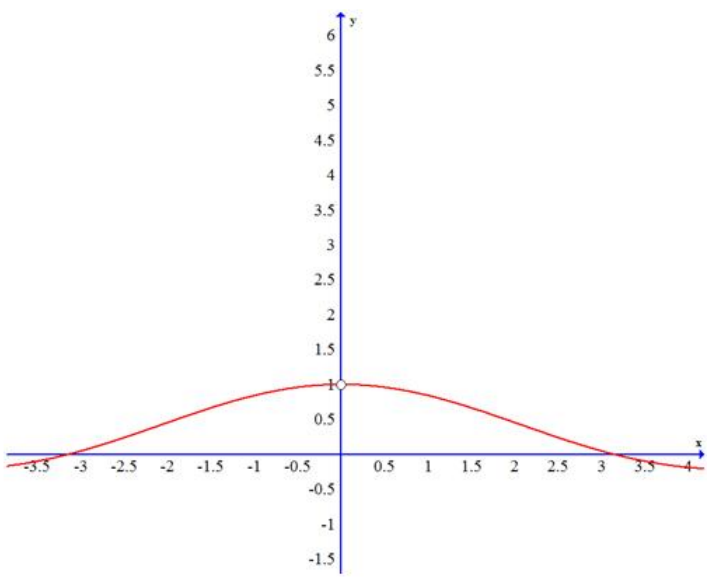
$$\lim_{x \to 0} 1 = 1$$

$$\lim_{x \to 0} \cos x = \cos 0 = 1$$

So, both functions approach 1 as $x \to 0$, and they squeeze the function in the middle! So, by the squeeze theorem:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Let's graph this function to visualize better what we are talking about:



This may seem like a lot of work to prove a simple limit, but we'll soon see the fruits of our effort. We will use it tomorrow to solve other limits involving trigonometric functions.

4 Still To Come

- **Day 6: Trigonometric Limits: These Can Be Tricky, Here Are All The Secrets**
- **Day 7: Limits at Infinity: Master The Basic Technique**