Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Limits

Day 6: Trigonometric Limits

By Pablo Antuna

Contents

1 Welcome

Welcome to **Day 6** of the **Intuitive Online Calculus Course!** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we'll apply what we learned yesterday to solve limits involving trigonometric functions.

2 A First Example

Today we'll learn a technique that is useful for solving almost any trigonometric limit. Our first example is:

$$\lim_{x \to 0} \frac{\tan x}{x}$$

The fundamental idea to solve these limits is to try to make appear the fundamental limit ($\frac{\sin x}{x}$) in the expression.

We can write this limit as:

$$\lim_{x \to 0} \frac{\sin x}{x} . \frac{1}{\cos x}$$

The first factor approaches one, because that is what we proved yesterday! The second factor approaches one because $\cos 0 = 1$. So:

$$\lim_{x \to 0} \frac{\sin x}{x} . \frac{1}{\cos x} = 1.1 = 1$$

This limit was easy to solve. This simple idea is what is used to solve almost any trigonometric limit.

Let's now try to solve the limit:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

Here we need to make use of some old trigonometric identities. It could be a good idea to memorize the first one of these:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

Now, making:

$$2\theta = x \Rightarrow \theta = \frac{x}{2}$$

$$2\sin^2(\frac{x}{2}) = 1 - \cos x$$

So, making use of this in our limit:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{x^2} = 2\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{x}.\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{x}$$

We now need a $\frac{x}{2}$ in our denominator to make use of our fundamental limit. Making:

$$y = \frac{x}{2} \Rightarrow x = 2y$$

$$2\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{x}.\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{x} = 2\lim_{y\to 0}\frac{\sin y}{2y}.\lim_{y\to 0}\frac{\sin y}{2y}$$

$$= 2.\frac{1}{2}.\frac{1}{2}.\lim_{y\to 0}\frac{\sin y}{y}.\lim_{y\to 0}\frac{\sin y}{y} = \frac{1}{2}.1.1 = \frac{1}{2}.$$

This is a considerably *hard* limit. This is something you may expect to find in an exam. So, to prepare you just for that, you can play with the following exercises. Have fun!

4 Exercises

Tomorrow you'll receive the answers to these problems.

Solve the following trigonometric limits:

1. $\lim_{x \to 0} \frac{\sin 4x}{x}$
2. $\lim_{x \to 0} \frac{1-\cos x}{x}$
3. $\lim_{x \to 0} \frac{\sin^2 \frac{x}{3}}{x^2}$

5 Still To Come

- **Day 7: Limits at Infinity: Master The Basic Technique**