Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Limits

Day 7: Limits at Infinity

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1 Welcome

Welcome to **Day 7** of the **Intuitive Online Calculus Course!**. The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on limits at infinity and the techniques for solving them.

2 Intuitive Idea of Infinity

We will now turn to another kind of limits involving polynomial functions, limits at infinity. These are limits where the variable x is not approaching any value, but it is growing in a consistent manner. We use the notation:

$$\lim_{x \to \infty} f(x)$$

What does this mean? As always, this limit may or may not exist. When it exists, it means that the function f is approaching a specific value, say L, when x grows without bounds. What "grow without bounds" mean?

It means that to get $f(x)$ close to L, we only need to make x sufficiently big.

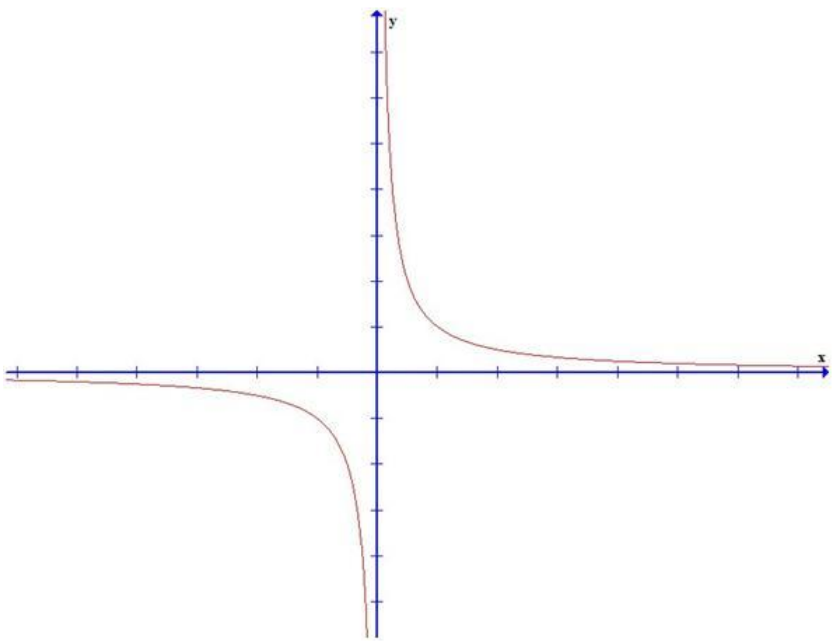
This concept may not be easy to digest at first, but an example may help. Let's consider the function:

$$f(x) = \frac{1}{x}$$

And the limit:

$$\lim_{x \to \infty} \frac{1}{x}$$

If we graph this function we'll get something like this:



We can see that as x grows, $\frac{1}{x}$ approaches 0. You may corroborate this with your calculator. Take any big number, like 1 billion, and divide 1 by it. You get very close to 0. So, in this case we say that:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Until now we only talked about positive infinity. Positive infinity? By this I mean that we only considered positive values of x when talking about the limit. We can also analyze a limit like this one:

$$\lim_{x \to -\infty} f(x)$$

This means that x becomes more and more negative. For example:

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

As it can be seen in the graph. We can also consider a limit when $x \to +\infty$. In the case of the function $\frac{1}{x}$, we use the simple notation ∞ to denote positive or negative infinity because it doesn't make difference (in this case!).

3 Calculating Limits at Infinity

How do we treat more complicated limits? First of all, there are some basic limits you should know:

$$\lim_{x\to\infty} \frac{a}{x} = 0, \qquad \lim_{x\to\infty} \frac{a}{x^n} = 0, \quad n \geq 1$$

This is just common sense. If you take a constant a and divide it by a big number, you get close to 0. If you take a constant a and divide it by a big number raised to the nth power you get close to 0 also. These may seem simple but we'll be using them a lot.

Now, a more interesting example...

4 The Basic Technique

I hope you got an intuition of what is a limit at infinity. But how do you go solving more complicated limits? Let's look at this example:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 1}{3x^3 - 5}$$

We cannot plug infinity and we cannot factor. So, now we'll use the basic technique used to solve almost any limit at infinity. It is a little algebraic trick.

Remember the property of fractions that said that you can divide both the numerator and denominator by the same number and the fraction remains the same? We'll be using that.

We will divide this limit by x to the greatest power found in the function. In this case it is x^3. So, let's divide both the numerator and denominator by x^3:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 1}{3x^3 - 5} = \lim_{x\to\infty} \frac{\frac{4x^3}{x^3} - \frac{2x^2}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} - \frac{5}{x^3}}$$

Performing the pertinent simplifications:

$$\lim_{x\to\infty} \frac{\frac{4x^3}{x^3} - \frac{2x^2}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} - \frac{5}{x^3}} = \lim_{x\to\infty} \frac{4 - \frac{2}{x} + \frac{1}{x^3}}{3 - \frac{5}{x^3}}$$

Now we use the property of the quotient:

$$\lim_{x\to\infty} \frac{4 - \frac{2}{x} + \frac{1}{x^3}}{3 - \frac{5}{x^3}} = \frac{\lim_{x\to\infty}\left(4 - \frac{2}{x} + \frac{1}{x^3}\right)}{\lim_{x\to\infty}\left(3 - \frac{5}{x^3}\right)}$$

As x approaches infinity, all the number divided by x to any power will approach zero. So, we're left with:

$$\frac{\lim_{x\to\infty}\left(4 - \frac{2}{x} + \frac{1}{x^3}\right)}{\lim_{x\to\infty}\left(3 - \frac{5}{x^3}\right)} = \frac{4 - 0 + 0}{3 - 0} = \frac{4}{3}$$

5 Other Simple Technique

In these examples we won't be using the basic technique of dividing by the greatest power of x. We'll be using something even more basic. Suppose we have this limit:

$$\lim_{x\to\infty} \frac{x+1}{x}$$

We can *separate* this fraction:

$$\lim_{x\to\infty} \left(\frac{x}{x} + \frac{1}{x}\right) = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)$$

We did the reverse of adding fractions. Try to add these two fractions and you'll get the original function. Whenever you have two or more terms in the numerator, and only one term in the denominator, you may try to do this.

You simply put each term in the numerator divided by the denominator and add them.

Now, we know that $1/x$ approaches zero when x approaches infinity. So, we have:

$$\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x} = 1 + 0 = 1.$$

That was straightforward!

6 Limit at Infinity With Radical

What if we have radicals in our function:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$$

This may seem different at first, but we only need to apply the same technique. In this case you may be tempted to divide everything by x^2, but notice that if you do so the denominator will go to 0, and that is no good.

So, we divide by x:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{\frac{x+1}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{1+\frac{1}{x}}$$

Now, we need to consider two cases: when $x \to +\infty$ and when $x \to -\infty$. Why? Because we have a square root, and we can only take the square root of a positive number. This implies that we need to take special care when $x \to -\infty$.

In the case $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{1+\frac{1}{x}} = \frac{\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x}}{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)} = \frac{\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x}}{1} = \lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x}$$

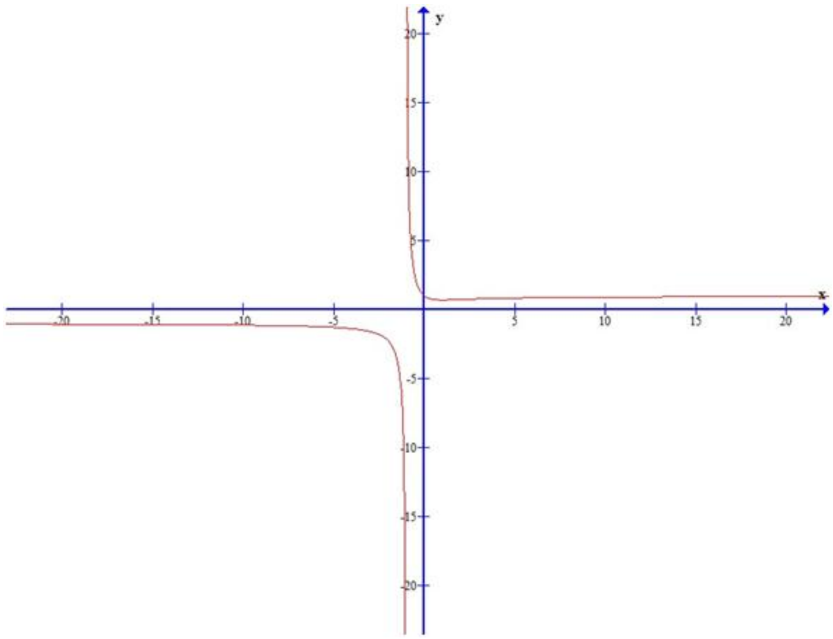
Now we'll do the following trick: we'll insert the denominator inside the radical, in the following way:

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to+\infty} \sqrt{\frac{x^2+1}{x^2}} = \lim_{x\to+\infty} \sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to+\infty} \sqrt{1+\frac{1}{x^2}}$$

And now we'll use the property of the square root that lets us plug the value into the root (continuity):

$$\lim_{x\to+\infty} \sqrt{1+\frac{1}{x^2}} = \sqrt{\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)} = \sqrt{1} = 1.$$

A graph of this function can help visualize this:



We can see that the function in fact approaches 1 when $x \to +\infty$. We can also see that the function approaches -1 when $x \to -\infty$. How can we prove this analitically? Let's give it a try, we want to calculate the limit:

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$$

We will want to repeat the trick of inserting an x into the root sign, but we can only do this if x is positive. When $x \to -\infty$, x is negative. But $-x$ is positive! So, we divide everything by $-x$:

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+1}}{-x}}{\frac{x+1}{-x}} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+1}}{-x}}{-1+\frac{1}{-x}}$$

Now, the denominator will go to -1 when $x \to -\infty$, so we're left with:

$$\lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+1}}{-x}}{-1} = -\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{-x}$$

We know that $-x$ is positve, so we can move it inside the square root:

$$-\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{-x} = -\lim_{x\to-\infty} \sqrt{\frac{x^2}{(-x)^2} + \frac{1}{(-x)^2}} = -\lim_{x\to-\infty} \sqrt{1 + \frac{1}{(-x)^2}}$$

And now we use the continuity of the square root:

$$-\lim_{x\to-\infty}\sqrt{1+\frac{1}{(-x)^2}} = -\sqrt{\lim_{x\to-\infty}\left(1+\frac{1}{(-x)^2}\right)} = -\sqrt{1} = -1.$$

And this agrees with our graph. This is a very good example, and you should try to review it if there's something that wasn't clear enough for you.

7 Exercises

1. $\lim_{x\to\infty}\left(2-\frac{1}{x}+\frac{4}{x^2}\right)$
2. $\lim_{x\to\infty}\frac{3x^2-2x-1}{x^3+4}$
3. $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$