Intuitive-Calculus.com Presents

The Free Intuitive Calculus Course

Derivatives

Day 9: Physical Interpretation of the Derivative

By Pablo Antuna

Contents

1 Welcome

Welcome to **Day 9** of the ***Intuitive Online Calculus Course!*** The main purpose of this course is to give you the basic tools to succeed in calculus, whether you're in high school, college or self-studying calculus!

Today we focus on the physical interpretation of the concept of derivative.

2 The Mechanical Interpretation of Slope

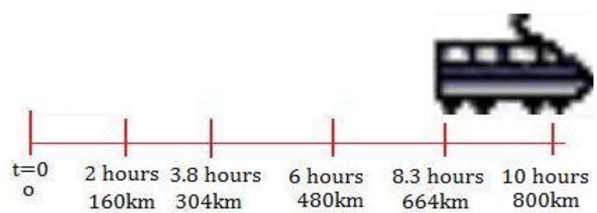
The conclusion we arrived yesterday was that the derivative gives us the slope of the tangent line to the curve.

You may say, *O.K., I can find the slope of the tangent line, and what else can the derivative do for me?*

At this point it may be difficult to see the connection between the definition of the derivative and something physical. However, it was physics what inspired this idea.

Consider a train travelling at constant velocity, say, 80km per hour.

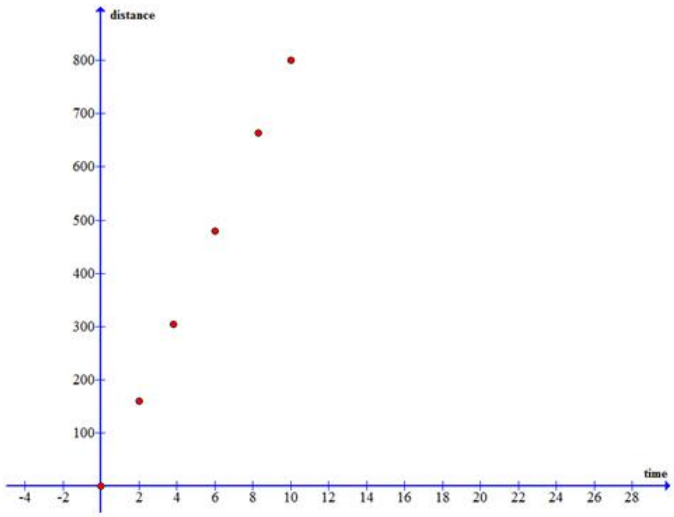
If at specific times during its travel we wrote down what distance it traveled we'd get a table like this one:



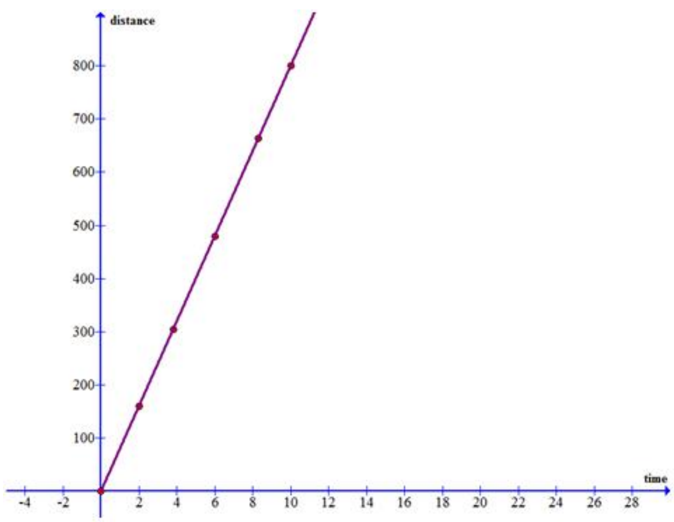
Time	Distance
0	0
2 hours	160km
3.8 hours	304km
6 hours	480km
8.3 hours	664km
10 hours	800km

We can see that distance is a function of time. So, we can make a graph, where the horizontal axis is for time and the vertical axis is for distance.

We can plot the points in a graph:



If we draw a curve passing through those points, we get a straight line:



Now, we know how to find the slope of this line. We take two points and find Δy and Δx. The slope is:

$$m = \frac{\Delta y}{\Delta x}$$

Why is this concept of slope useful? Remember that in our horizontal axis we represent time. That means that Δx is in fact time. Specifically, it is the time passed between the two points we choose.

We can write it as Δt instead of Δx to make it more explicit.

And also, in the vertical axis we represent distance. So, we can write Δs instead of Δy to denote distance.

This means that the slope is:

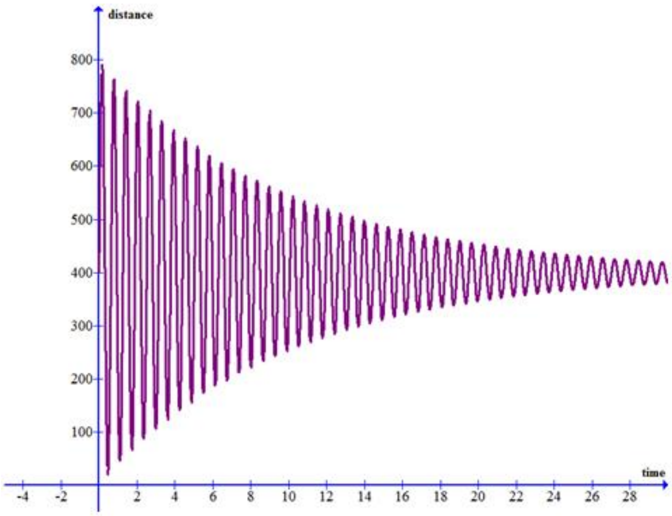
$$m = \frac{\Delta s}{\Delta t}$$

The slope is distance over time. You may recall from physics that this is how we measure velocity. So, the slope of the line is just the velocity of the train.

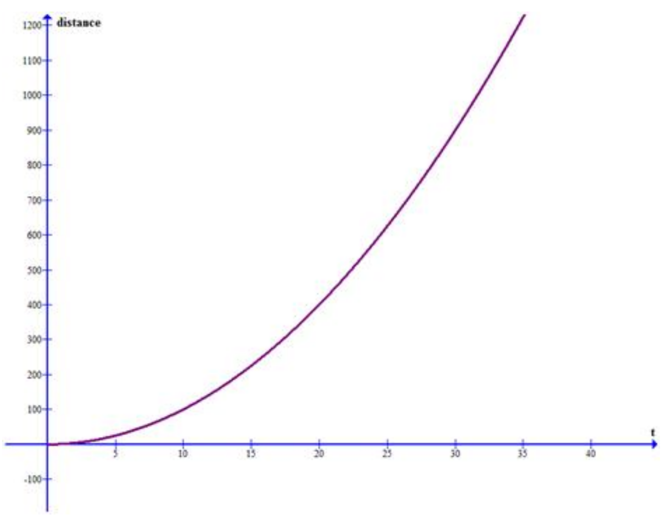
3 Now With Acceleration

Because the train was traveling at constant velocity, we got a line as the graph for distance. In reality, most things don't travel at constant velocities. Most cars accelerate and decelerate continuously, for example.

If we tried to graph distance for something that accelerates and decelerates we may get a crazy curve like this one:



Don't worry, though. We'll first look at a much simpler one. For example, let's say that the distance a car travels over time is represented by this graph:



Yesterday we learned that we can take two points on the graph of a function and calculate the slope of the secant line between those two points. That is, we can calculate the slope $\frac{\Delta y}{\Delta x}$ of the secant line.

The fraction $\frac{\Delta y}{\Delta x}$ represents a velocity, because it is distance over time. But this is just an average velocity between two points. This is because we considered the distances and times of only two points.

What if we wanted to know the velocity exactly at a specific point A? This is the type of velocity we usually talk about in our every day lives. And here is when the derivative comes into play.

We want the *velocity at a specific instant.* So, what we can do is to approximate by taking average velocities. An approximate average velocity will be the velocity over a very small time interval.

We can take smaller and smaller time intervals. In the language of calculus, that means that we take the limit as Δx approaches 0.

When we take smaller and smaller Δx's what happens is something we've already seen: we end up with the slope of the tangent line: See an animation of this.

So, it is natural to define the *instantaneous velocity* at an instant t_0 as the limit:

$$\text{speed} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

And this is exactly the definition of the derivative we know. What a surprise!

This instantaneous velocity is just a theoretical concept. When we measure velocities in reality we're just measuring average velocities over very small time intervals.

This is not an observable quantity. However, it allows us to talk precisely about the velocity at a specific point or the velocity at a specific instant.

Is the derivative only useful for finding velocities? No. There are many quantities defined in physics that are *rates of change*.

The basic example is velocity: the rate of change of distance over time. As a rule of thumb, a rate of change over time is the change in the quantity over the change in time.

Another simple example in physics is electric current. Current is defined as the rate of change of electric charge over time. So, using calculus, instantaneous current is the derivative of charge with respect to time.

So, we conclude that the derivative is *instantaneous rate of change*

Tomorrow we'll learn how to use the definition of the derivative to find the derivatives of some actual functions.

4 Still To Come

- **Day 3: Finding Derivatives by Applying the Definition**
- **Day 4: Derivatives of Trigonometric Functions**
- **Day 5: The Chain Rule**
- **Day 6: The Product Rule and Quotient Rule**
- **Day 7: Implicit Differentiation and Inverse Trigonometric Functions**